

# L-like models with large cardinals and a quasi lower bound on the consistency strength of PFA

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## L-like

Vaguely speaking, for a model of set theory to be L-like means that it satisfies properties of Gödel's constructible universe of sets  $L$ .

## Examples of L-like models

- $L$
- Fine Structural Inner Models
  - $L[U]$ , the canonical inner model for a measurable cardinal
  - Extender models of the form  $L[\vec{E}]$
- L-like Outer Models

Of course,  $L$  itself is as L-like as possible. Why are we looking for other L-like models at all?

Because  $L$  does not allow for the larger large cardinals.

# The Consistency Strength of BPFA

L-like models are very useful to determine the consistency strength of set theoretic principles. We will give an example, using the principle BPFA (the Bounded Proper Forcing Axiom), which is a strengthening of Martin's Axiom (for  $\aleph_1$ ):

## Theorem (Goldstern-Shelah, 1995)

- 1 *If there is a reflecting cardinal, then BPFA holds in a proper forcing extension of the universe.*
- 2 *If BPFA holds, then  $\omega_2$  is reflecting in  $L$ .*

This theorem shows that the consistency strength of BPFA is exactly that of a reflecting cardinal.

The Proper Forcing Axiom (PFA) is a significant strengthening of Martin's Axiom (for  $\aleph_1$ ) that has many applications in set theory but also outside of set theory. While Martin's Axiom can be obtained by forcing over any model of ZFC (and thus is equiconsistent with ZFC alone), PFA has much higher consistency strength. A consistency upper bound is given by the following classic theorem:

### Theorem (Baumgartner, 1984)

*If there is a supercompact cardinal, then PFA holds in a proper forcing extension of the universe.*

# A lower bound for PFA

It is generally conjectured that the consistency strength of PFA actually is that of a supercompact cardinal, i.e. if PFA holds, then there should be a model with a supercompact cardinal. The problem with actually verifying the above conjecture is that no inner models that can contain large cardinals even remotely in the range of a supercompact can be constructed to date - the best possible at the moment is Woodin limits of Woodin cardinals. Therefore consistency lower bounds beyond Woodin cardinals seem currently out of reach.

We want to introduce the idea of a quasi lower bound and use this idea together with L-like outer models to verify a quasi lower bound result on the consistency strength of PFA.

## L-like principles

- GCH
- $\diamond$  ( $\diamond_\kappa$  for every regular, uncountable  $\kappa$ )
- $\square_\kappa$  for various  $\kappa$ , global  $\square$
- Definable wellorder of the universe
- Condensation

## The Outer Model Programme

Basic Idea: Starting from a model of ZFC with large cardinals, obtain L-like properties in a forcing extension and preserve large cardinals.

## Theorem (Friedman, 2007)

- $\text{Con}(\omega\text{-superstrong}) \rightarrow \text{Con}(GCH + \omega\text{-superstrong})$
- $\text{Con}(\omega\text{-superstrong}) \rightarrow \text{Con}(\diamond + \omega\text{-superstrong})$
- $\text{Con}(\omega\text{-superstrong}) \rightarrow \text{Con}(\text{def. wo.} + \omega\text{-superstrong})$

For  $\square$ , the situation is slightly more complicated, as large cardinals impose restrictions on its validity:

## Limitations for $\square$

- If  $\kappa$  is subcompact,  $\square_\kappa$  fails. (Jensen)
- If  $\kappa$  is supercompact,  $\square_\lambda$  fails for every  $\lambda \geq \kappa$ . (Solovay)

## $\square$ in the Outer Model Programme

While stronger results have been obtained, we will only mention and make use of the following:

### Theorem (folklore, Cummings, Friedman)

*Given a model of ZFC, there is a forcing extension which preserves many large cardinals (in particular all subcompacts) and satisfies the following L-like properties:*

- *GCH*
- $\square_\lambda$  for every singular  $\lambda$
- $\square$  on the singular cardinals

But for our intended application on the consistency strength of PFA, we need an even more L-like model...



## Theorem (Gödel, 1939)

If  $M \prec (L_\alpha, \in)$ , then for some  $\bar{\alpha} \leq \alpha$ ,  $M \cong (L_{\bar{\alpha}}, \in)$ .

We want to obtain some amount of Condensation in our L-like model. In contrast to the other L-like principles considered so far, we first have to clarify what the statement of our condensation principle is supposed to be when taken out of the context of L:

## Models of the form $L[A]$

To define our desired Condensation property, we will assume that we are in a model of the form  $V=L[A]$  where  $A$  is a function from  $\text{Ord} \times \text{Ord} \rightarrow \text{Ord}$  such that for any ordinal  $\alpha$ ,  $\text{range}(A \upharpoonright (\alpha \times \alpha)) \subseteq \alpha$ .

If  $M$  is a substructure of  $(L_\alpha[A], \in, A)$ , we say that  $M$  *condenses* if for some  $\bar{\alpha} \leq \alpha$ ,  $M \cong (L_{\bar{\alpha}}[A], \in, A)$ .

# Generalized Condensation Principles

## Local Club Condensation

Assume  $V = L[A]$ . If  $\alpha$  has uncountable cardinality  $\kappa$  and  $\mathcal{A} = (L_\alpha[A], \in, A, \dots)$  is a structure for a countable language, then there exists a continuous chain  $\langle \mathcal{B}_\gamma : \omega \leq \gamma < \kappa \rangle$  of condensing substructures of  $\mathcal{A}$  whose domains  $B_\gamma$  have union  $L_\alpha[A]$ , each  $B_\gamma$  has cardinality  $\text{card } \gamma$  and contains  $\gamma$  as a subset.

But we need yet another property to hold, which follows easily from Condensation in L (but not from Local Club Condensation):

## Acceptability

Assume  $V=L[A]$ . For any ordinals  $\gamma \geq \delta$ , if there is a new subset of  $\delta$  in  $L_{\gamma+1}[A]$ , then

$$H^{L_{\gamma+1}[A]}(\delta) = L_{\gamma+1}[A].$$

We say that  $A$  witnesses Local Club Condensation or Acceptability respectively in the above.

## Theorem (Friedman, H)

*Starting with a model containing an  $\omega$ -superstrong cardinal  $\kappa$ , we can perform a cofinality-preserving class forcing to obtain a generic extension of the form  $L[A]$  such that  $A$  witnesses both Local Club Condensation and Acceptability and the  $\omega$ -superstrength of  $\kappa$  is preserved.*

By cofinality-preservation, we will preserve any previously obtained instances of square and have thus obtained our desired L-like model  $L[A]$  which satisfies the following:

- GCH
- $\square_\lambda$  for every singular  $\lambda$ ,  $\square$  on the singular cardinals
- $A$  witnesses Local Club Condensation and Acceptability

Starting with a proper class of subcompacts, we may preserve those.

# Consistency of Fragments of PFA

## Definition

A notion of forcing  $P$  is said to be  $\lambda$ -linked if there is a function  $f: P \rightarrow \lambda$  such that two conditions  $p_0$  and  $p_1$  in  $P$  are compatible whenever  $f(p_0) = f(p_1)$ .

## Theorem (Neeman, Schimmerling)

*PFA for  $2^{\aleph_0}$ -linked forcing is equiconsistent with a  $\Sigma_1^2$ -indescribable cardinal. A  $\Sigma_1^2$ -indescribable gap  $[\kappa, \kappa^+)$  gives rise to a model of PFA for  $(2^{\aleph_0})^+$ -linked forcing.*

$\Sigma_1^2$ -indescribable gaps  $[\kappa, \kappa^+)$  are just slightly larger than subcompacts - they are subcompact limits of subcompacts (and a little more).

## Theorem (Neeman)

*Assume  $V$  is a proper (forcing) extension of a fine structural model  $M$  and satisfies PFA for  $(2^{\aleph_0})^+$ -linked forcing. Then there is a  $\Sigma_1^2$  indescribable gap  $[\kappa, \kappa^+)$  in  $M$ .*

The problem with this theorem is that no fine structural models with subcompacts are currently known to exist. Our L-like model is not fine structural, but luckily, Neeman's proof can be slightly adapted to work for our L-like model and we get the following:

## Theorem (Friedman, H)

*Assume  $V$  is a proper (forcing) extension of an L-like model  $M$  and satisfies PFA for  $(2^{\aleph_0})^+$ -linked forcing. Then there is a  $\Sigma_1^2$  indescribable gap  $[\kappa, \kappa^+)$  in  $M$ .*

# Our quasi lower bound result

By basically taking the contraposition of the last theorem, we obtain the following:

## Theorem (Friedman, H)

*It is consistent that there is a model with a proper class of subcompacts but no proper (forcing) extension satisfies PFA for  $(2^{\aleph_0})^+$ -linked forcing.*

It may be worth noting that it is not the L-likeness of our model that keeps us from forcing PFA as by our results it is possible to obtain L-like models with supercompact cardinals. What keeps us from forcing PFA hence is the lack of sufficiently large cardinals. Rephrasing the above, we might say:

A proper class of subcompacts is a quasi lower bound for PFA for  $(2^{\aleph_0})^+$ -linked forcing with respect to proper (forcing) extensions.

## Definition

We say that a large cardinal property is *local* if it is preserved by sufficiently distributive forcing.

The next fact illustrates that our quasi lower bound result does not give a nontrivial result about PFA in a straightforward way:

## Fact

*It is consistent to have a proper class of (any kind of) local large cardinals but no set forcing extension satisfies PFA.*

*Proof:* Starting over a given model with a proper class of local large cardinals, force  $\square_\kappa$  for a proper class of cardinals  $\kappa$  while preserving a proper class of the given local large cardinals. As PFA implies that  $\square_\kappa$  fails for every  $\kappa \geq \omega_2$ , former cannot hold in any further set-generic extension.

Our quasi-lower bound result however does not only talk about set-forcing but also shows that subcompacts are in general not enough to obtain PFA by class-forcing. That this is a nontrivial result is affirmed by the following:

### Lemma (H, 2013)

*Given a supercompact cardinal, it is possible to obtain a model with a proper class of subcompacts such that no set forcing extension satisfies PFA but there is a class forcing extension that satisfies PFA.*

Moreover any global large cardinal that exists in  $V_\kappa$  when  $\kappa$  is subcompact would also work as a quasi lower bound, as L-likeness is preserved under taking initial segments of the universe:

Any large cardinal existing in  $V_\kappa$  when  $\kappa$  is subcompact is a quasi lower bound for PFA with respect to proper (forcing) extensions.



Using completely unrelated techniques, Matteo Viale and Christoph Weiß were able to obtain related, but incompatible results:

### Theorem (Viale, Weiß, 2011)

- *Suppose  $\kappa$  is inaccessible and PFA is forced by a standard iteration of length  $\kappa$  that collapses  $\kappa$  to  $\omega_2$ . Then  $\kappa$  is strongly compact.*
- *Suppose  $\kappa$  is inaccessible and PFA is forced by a proper standard iteration of length  $\kappa$  that collapses  $\kappa$  to  $\omega_2$ . Then  $\kappa$  is supercompact.*

Thus Viale and Weiss obtain higher consistency quasi lower bounds, but only with respect to the narrower class of (proper) standard iterations (in fact, what they really need are certain covering and approximation properties, which are satisfied by standard iterations, to hold between the ground model and the forcing extensions).

All of the above make it very plausible that the consistency strength of PFA should be some global (=non-local) large cardinal property. Examples of such a property are:

## Global large cardinal properties

- Strong cardinals
- Strongly compact cardinals
- Supercompact cardinals

## Question

Are there any *nice* global large cardinal properties between Woodin and strongly compact cardinals?

## Open Questions 2/2

Neeman and Schimmerling showed that larger  $\Sigma_1^2$ -indescribable gaps  $[\kappa, \kappa^{+n})$  give models of larger fragments of PFA.

### Question

Is it possible to extend our quasi-lower bound result to larger fragments of PFA?

This basically comes down to the question whether it is possible to force stronger Condensation principles by sufficiently nice forcing.

Is it possible to turn our quasi-lower bound result into an actual lower bound result?

### Question

Does the consistency of PFA for  $(2^{\aleph_0})^+$ -linked forcing imply the consistency of a subcompact cardinal or a  $\Sigma_1^2$ -indescribable gap  $[\kappa, \kappa^+)$ ?

Thank you.