Well-founded sets

Exercise 1 *Assume f is a function from A to B with* $\operatorname{rank}(A) = \alpha$ *and* $\operatorname{rank}(B) = \beta$. *What can you say about* $\operatorname{rank}(f)$*? Argue that we can calculate* $\operatorname{rank}(f)$ *in the case when f is onto B (surjective).*

Exercise 2 *For which α is $|R(\alpha)| = \beth_\alpha$?*

Exercise 3 *If κ is strongly inaccessible, then $|R_\kappa| = \kappa$. Are there cardinals κ with this property which are not inaccessible?*

Exercise 4 *Find a set x so that $|x| < |\operatorname{trcl}(x)|$.*

Exercise 5 *Work in* ZFC *without the Axiom of Foundation. Show that every proper class $\mathbf{A}$ for which $x \subseteq \mathbf{A} \to x \in \mathbf{A}$ holds is a superclass of* $\mathbf{WF}$ *(in the obvious sense that $x \in \mathbf{WF}$ implies $x \in \mathbf{A}$).*

Hint: $\emptyset \subseteq \mathbf{A}$.

Definition 1 *A set x is called* heriditarily finite *(in german "hereditär endlich") iff $|\operatorname{trcl}(x)| < \omega$. H_ω denotes the set of all hereditarily finite sets.*

Exercise 6 *Show that $H_\omega = R(\omega)$.*

A simplified approach to exercise 3 from last time

Exercise 7 *Let λ be any infinite cardinal and show that*[1]

$$|\{X \subseteq \lambda \colon |X| = \lambda\}| = 2^\lambda.$$

Hint: Assume not and use that whenever $X \subseteq \lambda$ has size less than λ, its complement (within λ) has size λ.

Exercise 8 *Assume $\lambda \leq \kappa$ and show that there are κ^λ-many injective functions from λ to κ (i.e. $|\{f \in {}^\lambda\kappa \colon f \text{ injective }\}| = \kappa^\lambda$).*

Hint: Given $f \in {}^\lambda\kappa$, find a way to construct an injective $g_f \in {}^\lambda\kappa$ in an injective way, i.e. it should be the case that whenever $f_0 \neq f_1 \in {}^\lambda\kappa$, $g_{f_0} \neq g_{f_1}$.

Note: Almost the same proof shows that whenever λ is an infinite cardinal, there are 2^λ many bijections from λ to λ. We will need (and use) this in the following but I suggest to omit the exact proof.

[1] this is the special case where $\kappa = \lambda$ of exercise 3 from last time

Exercise 9 *Show that whenever* $\lambda \leq \kappa$ *are infinite ordinals,*[2]

$$|\{X \subseteq \kappa \colon |X| = \lambda\}| = \kappa^\lambda.$$

Hint: Let $A := \{X \subseteq \kappa \colon |X| = \lambda\}$. Argue that $|A| \leq \kappa^\lambda$ is pretty obvious. We have to show that $\kappa^\lambda \leq |A|$. We want to find an injection from ${}^\lambda\kappa$ to A. By exercise 8, it suffices to find an injection from the injective functions from λ to κ to A. Note that every injective $f \colon \lambda \to \kappa$ is naturally connected to an element of A, namely to range(f). As there are 2^λ many bijections from λ to λ (see note above), 2^λ-many functions functions f are connected to the same element of A for each element of A. This gives rise to the equation $|A| \otimes 2^\lambda = \kappa^\lambda$. Use this to obtain the desired result distinguishing the cases $2^\lambda < \kappa^\lambda$ and $2^\lambda = \kappa^\lambda$.[3]

[2] this is exactly exercise 3 from last time

[3] I apologize for posing this problem in the last exercise without any hints or initial steps, it seems too hard - but maybe there's an easier solution?