Set Theory Reading Course, Winter 2011 / 2012

Week 1, October 4th, 2011:

Please read (about 10 pages total, best in the order given below, everything refers to Kunen's Set Theory book):

- VII, 1 - General Remarks
- II, 2 from definition 2.1 until (including) definition 2.4. Examples 3 and 4 on page 53 as well as the paragraph between example 4 and definition 2.4 may be ignored.
- VII, 2 - Generic Extensions; ignore several small references to MA and to chapter II.

Exercises: A1, A4, A5 from the Exercises of Chapter VII, which may be found at the very end of the chapter. Hint for A4: this is very similar to / is an extension of the proof of VII, Lemma 2.3.

Typos:

- II, 2, page 53, definition 2.2: $\cup$ should be replaced by $\vee$.
- II, 2, page 53, very last line: $q \leq p$ should be replaced by $p \leq q$.

Week 2, October 11th, 2011:

Please read (about 11 pages total, best in the order given below):

- II, 2 from definition 2.5 (MA) until (including) the paragraph after example 6 on the next page.
- VII, 3 - Forcing.

Remarks:

- Definition 3.3 ($\Vdash^*$), while quite natural once one is familiar with forcing, may seem to come out of nowhere at the moment. Kunen tries to motivate this definition after he gives it in his book. The exact definition of $\Vdash^*$ will never be needed after §3.
- The proof of Theorem 3.5 might be relatively hard to read while you learn forcing and becomes quite obvious once familiar with the forcing technique. While the theorem itself will be of great importance for us, the exact proof won't (which doesn't mean you shouldn't read it).

Exercises:

- Exercise A9 from VII.
- Proof Lemma 3.2 (using "only" definition 3.1, don't use $\Vdash^*$).
- Proof what Kunen claims to be "instructive to check" on page 195.

Typos:

- VII, 3, page 193: in the second line f should be replaced by f_G.
- VII, 3, page 198: "We must show that $\pi_{1G} \in \tau_{2G}$." should be replaced by "We must show that if $s_1 \in G$, then $\pi_{1G} \in \tau_{2G}$.".

Week 4, October 25th, 2011:

Please read (about 13 pages total, 2 weeks, you may also read II, 1 before VII, 4 if you like):

- VII, 4 - ZFC in $M[G]$
- II, 1 - Almost disjoint and quasi-disjoint sets until (including) the proof of Theorem 1.6.
- VII, 5 - Forcing with finite partial functions

Remarks:

- While we read the page about almost disjoint sets in II rather for cultural enrichment than because we will need it later in this course (although they are important for forcing, but we probably won't get to the point of using them), Theorem 1.6 about delta-systems will soon become important for the development of our first application of forcing, namely to show that the negation of CH is consistent with ZFC (assuming the consistency of ZFC of course) in VII, 5.

Exercises: A12 from the exercises of chapter VII: show only that (a) and (b) are equivalent.

Typos:

- II, 1: being pedantic, one should replace "then $|\mathcal{A}| = 2^\kappa$ and is an a.d. family" by "then $|\mathcal{A}| = 2^\kappa$ and $\mathcal{A}$ is an a.d. family" in the proof of Theorem 1.3.
- VII, 5: In the proof of Lemma 5.5, b is missing a check once: $r \Vdash \tau(\check{a}) = b$ should in fact be $r \Vdash \tau(\check{a}) = \check{b}$.

Week 5, November 8th, 2011:

Please read (about 7 pages) VII, 6 - Forcing with partial functions of larger cardinality.

Preview for the rest of the semester: I tried to fix an approximate plan for what we should read for the rest of the semester:

- parts of Kunen, VII, 7 - Embeddings, Isomorphisms, ignoring Boolean-valued models (5 pages)
- parts of Kunen, VII, 8 - Further Results: 8.1, 8.2, 8.5-8.8 (2 pages)
- Iterated Forcing from "Thomas Jech: Multiple Forcing", Cambridge Tracts in Mathematics 88, 1986, part II: Iterated forcing, sections 1–4 (pages 44–58 = 15 pages)
- Clubs and Stationary sets from "Thomas Jech: Set Theory, Third Millennium Edition", Springer, 2006, chapter 8: Stationary sets until Lemma 8.11 (6 pages)
- Measurable Cardinals from "Thomas Jech: Set Theory, Third Millennium Edition", chapter 17: Large Cardinals until Lemma 17.6 (5 pages)
- Maybe one more thing - an obvious candidate would be countable support iterations from Jech's "Multiple Forcing" book...

Week 6, November 15th, 2011:

Please read (about 7 pages) the following parts of VII, 7 and VII, 8, ignoring any references to Boolean-valued models throughout:

- from the beginning of VII, 7 to (including) Lemma 7.13
- VII, 8: Lemma 8.1, Theorem 8.2 and the paragraph in-between them
- VII, 8: from Lemma 8.5 (and the small paragraph before) until the end of page 231 (don't bother too much about the Solovay example mentioned at the end of page 231; in this course we will just have this mentioned to illustrate what one can possibly do by "just" collapsing cardinals using the Lévy collapse).

Week 7, November 22nd, 2011:

Please read (about 6 pages) Chapter 8 from Jech's Set Theory book until and including Lemma 8.11 (and its proof), ignoring the 8-line paragraph about the quotient algebra in the middle of page 93.

Typo: In the paragraph following the proof of Lemma 8.8, the second sentence should rather be something like this: Every stationary subset of E^{κ}_{λ} is the union of κ disjoint stationary sets.

Week 8, November 29th, 2011:

Please read the following parts of part II of Thomas Jech's book "Multiple Forcing" (about 10 pages) in the next 2 weeks:

- from the beginning of part II until (including) Formula (1.4).
- from the sentence before Proposition 1.9 until (including) Lemma 1.11, ignoring clause (b) everywhere (Proposition 1.9, Theorem 1.10, Lemma 1.11).
- Chapter 2 (Finite support iteration), ignoring three-and-a-half lines after formula (2.2).
- Chapter 3 (Martin's Axiom)
- Chapter 4 (Suslin's Problem)

Notation:

- Instead of using $\tau, \sigma, \pi, \ldots$ to denote names, they are usually denoted by symbols with a dot above, like $\dot{x}$. Usually, if x is an object in the generic extension, $\dot{x}$ denotes a name for x and vice versa if $\dot{x}$ is a name, x denotes $\dot{x}_G$, the evaluation of $\dot{x}$ by a generic G.
- Instead of writing $\dot{x}_G$ for the evaluation of $\dot{x}$ by the generic filter G, Jech writes $\dot{x}/G$.
- Jech just writes 1 for the weakest condition $\mathbf{1}$ of a forcing.
- $\|\varphi(\dot{x})\|_P = 1$ stands for $1 \Vdash_P \varphi(\dot{x})$.[1]
- $1 \Vdash_P \varphi$ is usually abbreviated by $\Vdash_P \varphi$.
- V^P denotes the class of P-names in V.
- $p|q$ means that p and q are compatible (conditions in some forcing P)

[1]This notation comes from Boolean-valued models which Jech uses to introduce forcing unlike Kunen, who uses countable transitive models of (fragments of) ZFC. As we still want to avoid boolean-valued models, we will have to do some translations.

We will need the following (I couldn't find it in Kunen's book):

Lemma 1: For any formula $\varphi(\dot{x})$, $D = \{p \in P \colon p \Vdash \varphi(\dot{x}) \vee p \Vdash \neg\varphi(\dot{x})\}$ is dense.

Proof Let $p \in P$ be given. In any P-generic extension with generic G, either $\varphi(\dot{x}/G)$ or $\neg\varphi(\dot{x}/G)$ holds. By Theorem 3.6, (2) from Kunen's book (chapter VII), it follows that there is a condition $q \in G$ which either forces $\varphi(\dot{x})$ or $\neg\varphi(\dot{x})$. As G is a filter, we may find a condition r in G which is stronger than both p and q. Then r forces $\varphi(\dot{x})$ by Lemma 3.4 from Kunen's book (chapter VII) and hence $r \in D$, showing that D is dense as p was arbitrary. □

We will also need the following, which I stated without proof in last week's lecture:

Lemma 2: If τ is a name for an element of V, then $\forall p \in P\, \exists r \leq p\, \exists x \in V$ such that $r \Vdash \tau = \check{x}$.

Proof Fix τ and p as above. Let G be P-generic (over V) such that $p \in G$. Let $x = \tau^G \in V$. By the Forcing Theorem (Kunen, Chapter VII, Theorem 3.6) there is $q \in G$ such that $q \Vdash \tau = \check{x}$. Since G is a filter, there is $r \in G$ which is stronger than both p and q. Then r also forces $\tau = \check{x}$ (Lemma 3.4, Kunen ch. VII). But then r and x are as desired. □

Remarks:

- In the proof of Lemma 1.3, (a) $D_1 = \{(p, \dot{q}) \colon p \in D\}$ could be written somewhat more exactly as

$$D_1 = \{(p, \dot{q}) \colon p \in D \wedge \| \dot{q} \in \dot{Q} \| = 1\}.$$

- The proof of Lemma 1.3, (b) starts out as follows:

  "Let $D \in V[G]$ be dense in Q and let $\dot{D} \in V^P$ be a name for D such that $\Vdash_P \dot{D}$ is dense in $\dot{Q}$."

  Why can we find such a name $\dot{D}$? Let's start out by taking any name σ for D (this just means that $\sigma/G = D$) and let A be a maximal antichain of P such that A is the disjoint union of B and C and:

  - for all $a \in B$, $a \Vdash \sigma$ is dense in $\dot{Q}$,
  - for all $a \in C$, $a \Vdash \sigma$ is not a dense subset of $\dot{Q}$.

  This can be done using Lemma 1 from above, which tells us that the set of conditions in P forcing either that σ is dense in $\dot{Q}$ or forcing that σ is not dense in $\dot{Q}$ is dense in P and then we use the fact that I proved last lecture - namely that any dense set contains (as a subset) a maximal antichain. This maximal antichain will be our A. Now we can obviously partition A into $B \cup C$ such that the above holds.

Now use Lemma 8.1 from Kunen's book (chapter VII) to get from σ to our desired name $\dot{D}$ as follows: if $a \in B$, let $\sigma_a = \sigma$; if $a \in C$, let $\sigma_a = \dot{Q}$. By Lemma 8.1, we may choose $\dot{D}$ such that $a \Vdash \dot{D} = \sigma_a$ for each $a \in A$. It is now easily seen that $1 \Vdash \dot{D}$ is dense in $\dot{Q}$ and that $\dot{D}/G = D$. For the former, note that $1 \Vdash \dot{Q}$ is dense in $\dot{Q}$.

- In the proof of Lemma 1.11 use Lemma 6.9 (Kunen, VII) instead of Theorem 2.14 in part I of Jech's book. It follows that whenever $\dot{Z}$ is a name for a subset of κ, $\Vdash \exists \gamma < \check{\kappa}\ \dot{Z} \subseteq \gamma$ (otherwise κ could not be forced to be regular). Now use Theorem 8.2 from Kunen's book to obtain a name $\dot{\gamma}$ so that $\Vdash \dot{Z} \subseteq \dot{\gamma} < \check{\kappa}$. Now $\dot{\gamma}$ is a name for a ground model object (namely an ordinal) and hence we can find a maximal antichain W such that $\forall p \in W\, \exists \gamma_p$ with $p \Vdash \dot{\gamma} = \check{\gamma}_p$ (we can find such an antichain because the set of such p is dense by Lemma 2 above). Note that if W is a maximal antichain of P, then W has size $< \kappa$ by the κ-cc. Hence $\gamma = \sup \gamma_p$ is less than κ and the trivial condition 1 forces that $\dot{\gamma}$ is at most γ and hence that $\dot{Z}$ is a subset of γ.

- Replace the proof of Theorem 1.10 (a) by the following:

  Assume that $(p_\alpha, \dot{q}_\alpha)$, $\alpha < \kappa$ are mutually incompatible. In $V[G]$ (for a V-generic G on P), let $Z = \{\alpha\colon p_\alpha \in G\}$. Note that $p_\alpha \Vdash \alpha \in \dot{Z}$. Let $\dot{Z}$ be a P-name for Z in V. For any α and β, if r is stronger than both p_α and p_β, then $r \Vdash \dot{q}_\alpha \perp \dot{q}_\beta$, as otherwise $(p_\alpha, \dot{q}_\alpha) \parallel (p_\beta, \dot{q}_\beta)$. Thus $q_\alpha \perp q_\beta$ whenever $\alpha, \beta \in Z$ (this is because elements of G are compatible to each other; $q_\alpha = \dot{q}_\alpha/G$, $q_\beta = \dot{q}_\beta/G$). Since $Q = \dot{Q}/G$ has the κ-cc in $V[G]$, we have $|Z| < \kappa$. By Lemma 1.11, there is $\gamma < \kappa$ such that $\Vdash \dot{Z} \subseteq \check{\gamma}$, but that contradicts the fact that $p_\gamma \Vdash \check{\gamma} \in \dot{Z}$. □

- Replace Lemma 2.4 and its proof by the following:

Lemma 2.4: If P_α is a finite support iteration and $P_\beta = P_\alpha \restriction \beta$, then there is a complete embedding i from P_β to P_α. Hence forcing with P_α yields larger generic extensions than forcing with P_β does.

Proof For $p \in P_\beta$, let $i(p) = q \in P_\alpha$ with q such that $q \restriction \beta = p$ and $q(\gamma) = 1$ whenever $\beta \leq \gamma < \alpha$. The final statement follows from Theorem 7.5 from Kunen's book, chapter VII. □

Week 9, December 13th, 2011

Please read:

- Chapter 7 (Filters, Ultrafilters and Boolean Algebras) from Thomas Jech: Set Theory until (and including) Theorem 7.5, it's proof and the two small two-line paragraphs afterwards.
- Chapter 12 (Models of Set Theory) from Thomas Jech: Set Theory:
  - Review of Model Theory
  - Reduced Products and Ultraproducts
- Chapter 17 (Large Cardinals) from Thomas Jech: Set Theory until the end of page 289.
- For those interested, I will also provide a copy of page 290, which contains a - very interesting - theorem by Kunen. I won't ask you about this at the exam though.
- I will also hand out copies of a proof of the consistency of ¬AC together with ZF. I won't ask you about this at the exam either.