SIMPLEST POSSIBLE LOCALLY DEFINABLE WELL-ORDERS

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Abstract. We study the provable consequences of the existence of a well-order of $\mathrm{H}(\kappa^+)$ definable by a Σ_1-formula over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ in the case where κ is an uncountable regular cardinal. This is accomplished by constructing partial orders that force the existence of such well-orders while preserving many structural features of the ground model. We will use these constructions to show that the existence of a well-order of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula with parameter ω_1 is consistent with a failure of the GCH at ω_1. Moreover, we will show that one can achieve this situation also in the presence of a measurable cardinal. In contrast, results of Woodin imply that the existence of such a well-order is incompatible with the existence of infinitely many Woodin cardinals with a measurable cardinal above them all.

1. Introduction

Given an infinite cardinal κ, if the set of all subsets of κ is constructible from some subset z of κ, then there is a well-order of the set $\mathrm{H}(\kappa^+)$ of all sets of hereditary cardinality at most κ that is *locally definable* by a Σ_1-formula with parameter z, i.e. there is such a well-order that is definable over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula[1] with parameter z. In particular, if the power set of κ is contained in Gödel's constructible universe L, then there is a well-order of $\mathrm{H}(\kappa^+)$ that is locally definable by a Σ_1-formula without parameters. One may view such well-orders as the *simplest possible locally definable well-orders of* $\mathrm{H}(\kappa^+)$, because their definition uses no parameters and a short argument shows that no formula that lies lower down in the Levy hierarchy defines a well-order over a structure of the form $\mathrm{H}(\kappa^+)$ for any infinite cardinal κ. For the sake of completeness, we present this argument.

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[1]Note that if $\lhd$ is a well-order, then $x \lhd y \iff [x \neq y \ \wedge\ \neg(y \lhd x)]$. Thus any Σ_1-definable well-order of $\mathrm{H}(\kappa^+)$ is in fact Δ_1-definable. Proposition 1.1 shows that a Σ_1-formula defining a well-order is not provably equivalent to the Π_1-formula obtained from the above equivalence.

Remember that a Σ_1-formula $\varphi(v_0,\ldots,v_{n-1})$ is $\Delta_1^{\mathrm{ZFC}^-}$ if there is a Π_1-formula $\psi(v_0,\ldots,v_{n-1})$ with

$$\mathrm{ZFC}^- \vdash \forall x_0,\ldots,x_{n-1}\ [\varphi(x_0,\ldots,x_{n-1}) \longleftrightarrow \psi(x_0,\ldots,x_{n-1})],$$

where ZFC^- denotes the axioms of ZFC without the Power Set Axiom. Note that every Σ_0-formula is a $\Delta_1^{\mathrm{ZFC}^-}$-formula.

Proposition 1.1. *Let κ be an infinite cardinal. Then there is no well-order of $\mathrm{H}(\kappa^+)$ that is definable over the structure $\langle \mathrm{H}(\kappa^+),\in\rangle$ by a $\Delta_1^{\mathrm{ZFC}^-}$-formula with parameters.*

Proof. Let $\varphi(v_0,v_1,v_2)$ be a $\Delta_1^{\mathrm{ZFC}^-}$-formula and let $\psi(v_0,v_1,v_2)$ be the corresponding Π_1-formula. Using the formula ψ, we can construct a Σ_1-formula $\Phi_0(v)$ with the property that the axioms of ZFC^- prove that for every set z, the statement $\Phi_0(z)$ is equivalent to the statement that the relation $\{\langle x,y\rangle \mid \varphi(x,y,z)\}$ is not linear. Moreover, we can use the formula φ to construct a Σ_1-formula $\Phi_1(v)$ with the property that the axioms of ZFC^- prove that for every set z, the statement $\Phi_1(z)$ is equivalent to the statement that the relation $\{\langle x,y\rangle \mid \varphi(x,y,z)\}$ is not well-founded. Set $\Phi(v) \equiv \Phi_0(v) \vee \Phi_1(v)$. By the Σ_1-Reflection Principle, the axioms of ZFC prove that whenever θ is an infinite cardinal and $z \in \mathrm{H}(\theta^+)$, then $\Phi(z)$ holds in $\langle \mathrm{H}(\theta^+),\in\rangle$ if and only if the set

$$\{\langle x,y\rangle \in \mathrm{H}(\theta^+)\times\mathrm{H}(\theta^+) \mid \langle \mathrm{H}(\theta^+),\in\rangle \models \varphi(x,y,z)\}$$

is not a well-order of $\mathrm{H}(\theta^+)$.

Assume, towards a contradiction, that there is an infinite cardinal κ such that the formula φ and some parameter $z \in \mathrm{H}(\kappa^+)$ define a well-order of $\mathrm{H}(\kappa^+)$ over the structure $\langle \mathrm{H}(\kappa^+),\in\rangle$. By the above remarks, this implies that $\neg\Phi(z)$ holds in $\langle \mathrm{H}(\kappa^+),\in\rangle$. Pick a regular cardinal $\nu > 2^\kappa$ and let G be $\mathrm{Add}(\nu,(\nu^{<\nu})^+)$-generic over V. Then $\nu = \nu^{<\nu}$ holds in $\mathrm{V}[G]$ and a folklore result (see, for example, [15, Corollary 9.2]) says that there is no well-order of $\mathrm{H}(\nu^+)^{\mathrm{V}[G]}$ that is definable over the structure $\langle \mathrm{H}(\nu^+)^{\mathrm{V}[G]},\in\rangle$. In particular, $\Phi(z)$ holds in $\langle \mathrm{H}(\nu^+)^{\mathrm{V}[G]},\in\rangle$ and Σ_1-reflection implies that it also holds in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]},\in\rangle$. But we have $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]} = \mathrm{H}(\kappa^+)^{\mathrm{V}}$, a contradiction. □

A classical theorem of Mansfield shows that the converse of the implication mentioned at the beginning of this section also holds in the case $\kappa=\omega$, in the sense that the existence of a locally Σ_1-definable well-order of $\mathrm{H}(\omega_1)$ implies that all subsets of ω are constructible from the parameters of this definition. In particular, the existence of such a well-order implies that CH holds and that there are no measurable cardinals.

Theorem 1.2 ([17]). *The following statements are equivalent for every $z \subseteq \omega$.*

(i) Every subset of ω is an element of $\mathrm{L}[z]$.
(ii) There is a well-order of the set $\mathrm{H}(\omega_1)$ that is definable over the structure $\langle \mathrm{H}(\omega_1), \in\rangle$ by a Σ_1-formula with parameter z.
(iii) There is a well-order of the set ${}^{\omega}\omega$ of all functions from ω to ω that is definable over the structure $\langle \mathrm{H}(\omega_1), \in\rangle$ by a Σ_1-formula with parameter z.

In this paper, we are interested in the provable consequences of the existence of locally definable well-orders of $\mathrm{H}(\kappa^+)$ of low complexity in the case where κ is an uncountable regular cardinal. In particular, we want to determine the *simplest definition* such that the existence of a well-order definable in this way is consistent together with certain natural set theoretical assumptions whose negation holds in L. Examples of such assumptions are failures of the GCH at κ or the existence of larger large cardinals above κ. For that purpose, we construct partial orders that force the existence of locally Σ_1-definable well-orders while preserving many structural features of the ground model. The starting point of this work is the following result proven in [11]. It can be used to show that many statements are compatible with the existence of locally Σ_1-definable well-orders if we allow arbitrary parameters in their definitions.

Theorem 1.3 ([11, Theorem 1.1]). *Let κ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$ holds*[2] *and 2^κ is regular. Then there is a partial order $\mathbb{P}$ with the following properties.*

(i) $\mathbb{P}$ is $<\kappa$-closed and forcing with $\mathbb{P}$ preserves cofinalities less than or equal to 2^κ and the value of 2^κ.
(ii) If G is $\mathbb{P}$-generic over the ground model V, then there is a well-order of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters.

The parameter used in the definition of this well-order is added by the forcing and therefore is, in a certain sense, a very complicated object. In this paper, we want to improve the above result by constructing models of set theory possessing locally Σ_1-definable well-orders that only use *simple* parameters. The first (and in fact main) step towards this goal will be the construction of a locally Σ_1-definable well-order in a generic extension whose definition only uses parameters that already exist in the ground model.

[2]Note that the assumption $\kappa = \kappa^{<\kappa}$ implies that κ is regular.

This is achieved (for a somewhat smaller class of cardinals κ) in the next theorem.[3] Remember that, given an uncountable cardinal κ, a subset S of κ is *fat stationary* if for every club $C \subseteq \kappa$, the intersection $C \cap S$ contains closed subsets of ordinals of arbitrarily large order-types below κ. Given regular cardinals $\eta < \kappa$, we let S^κ_η denote the set of all limit ordinals less than κ of cofinality η. The set $S^\kappa_{<\eta}$ is defined analogously.

Theorem 1.4. *Let κ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$, $\eta^{<\eta} < \kappa$ for every $\eta < \kappa$ and 2^κ is regular.*[4] *Assume that one of the following statements holds.*

(a) κ is the successor of a regular cardinal η and $\langle S_\alpha \mid \alpha \leq \kappa\rangle$ is a sequence of pairwise disjoint stationary subsets of S^κ_η.
(b) κ is an inaccessible cardinal and $\langle S_\alpha \mid \alpha \leq \kappa\rangle$ is a sequence of pairwise disjoint fat stationary subsets of κ.

Then there is a partial order $\mathbb{P}$ with the following properties.

(i) $\mathbb{P}$ is $<\kappa$-distributive and forcing with $\mathbb{P}$ preserves cofinalities less than or equal to 2^κ and the value of 2^κ.
(ii) If G is $\mathbb{P}$-generic over the ground model V, then there is a well-order of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameter $\langle S_\alpha \mid \alpha < \kappa\rangle$.

In the second part of this paper, we use Theorem 1.4 to construct models of set theory containing locally Σ_1-definable well-orders of some $\mathrm{H}(\kappa^+)$ that only use the ordinal κ as a parameter by forcing over certain canonical inner models of set theory. This will allow us to show that the existence of such a well-order is compatible with a failure of the GCH at κ. The following theorem is an example of such a construction.

Theorem 1.5. *Assume that $\mathrm{V} = \mathrm{L}$ holds and κ is either the successor of a regular cardinal or an inaccessible cardinal. Let $\mathbb{P}$ be a partial order with the following properties.*

(a) Forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and fat stationary subsets of κ.
(b) If G is $\mathbb{P}$-generic over V, then in $\mathrm{V}[G]$, 2^κ is regular, $\kappa = \kappa^{<\kappa}$ and $\eta^{<\eta} < \kappa$ for all $\eta < \kappa$.

[3]While the forcing construction in our paper is based on the construction in [11], one could instead base it on the construction that was later provided in [10]. This would eliminate the assumption that 2^κ be regular and would yield a forcing that preserves all cofinalities (rather than just those less than or equal to 2^κ) in our below results.

[4]Note that the second assumptions implies that κ is either an inaccessible cardinal or the successor of a regular cardinal.

Then there is a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order such that the following statements hold whenever $G * H$ is $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V*.*

(i) *The partial order $\dot{\mathbb{Q}}^G$ is $<\kappa$-distributive in* V$[G]$*.*
(ii) *Forcing with $\dot{\mathbb{Q}}^G$ over* V$[G]$ *preserves all cofinalities less than or equal to $(2^\kappa)^{\mathrm{V}[G]}$ and the value of 2^κ.*
(iii) *There is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}$ that is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}, \in\rangle$ by a Σ_1-formula with parameters κ.*

Letting $\mathbb{P} = \mathrm{Add}(\kappa, \kappa^{++})$, Theorem 1.5 shows that a failure of the GCH at an uncountable regular cardinal κ is consistent with the existence of a well-order of $\mathrm{H}(\kappa^+)$ that is locally definable by a Σ_1-formula with parameter κ.

By Mansfield's Theorem 1.2 and the Σ_1-Reflection Principle, the existence of a well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameter $z \in \mathrm{H}(\omega_1)$ implies that $\mathcal{P}(\omega) \subseteq \mathrm{L}[z]$. In particular, the assumption

$$(\star) \qquad \forall z \subseteq \omega \ \exists x \subseteq \omega \ x \notin \mathrm{L}[z]$$

implies that there is no such well-order. Note that $(\star)$ holds in all $\mathrm{Add}(\omega, \omega_1)$-generic extensions and that the partial order $\mathbb{P} = \mathrm{Add}(\omega, \omega_1)$ satisfies the requirements (a) and (b) of Theorem 1.5 in L. Since the partial order $\dot{\mathbb{Q}}^G$ from Theorem 1.5 adds no new reals whenever G is $\mathbb{P}$-generic over L, it preserves the statement $(\star)$. Thus Theorem 1.5 shows that in this setting, the above forcing construction for $\kappa = \omega_1$ adds a locally definable well-order of $\mathrm{H}(\omega_2)$ of the optimal complexity compatible with $(\star)$, in the sense of providing a Σ_1-definition with smallest possible parameter.

It is natural to ask whether it is possible to strengthen the above result and force the existence of well-orders of $\mathrm{H}(\omega_2)$ that are definable using smaller parameters. By the above remarks, the existence of such a well-order would imply the negation of $(\star)$. It is not known if such a well-order can exist outside of models of the form $\mathrm{L}[z]$ with $z \subseteq \omega$.

Question 1.6. *Does the existence of a well-order of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1)$ imply that $\mathcal{P}(\omega_1)$ is constructible from some subset z of ω?*

In another direction, one can ask whether assumptions like the ones listed in Theorem 1.5 are actually necessary for such constructions. In particular, it is interesting to ask whether the existence of such well-orders is compatible with the existence of larger large cardinals. A modification of the above construction yields the following result that shows that the existence of a well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameter κ is consistent with the existence of a measurable cardinal above κ and a failure of the GCH at κ.

Theorem 1.7. *Assume that U is a normal measure on a cardinal δ, $\mathrm{V} = \mathrm{L}[U]$ holds and $\kappa < \delta$ is either the successor of a regular cardinal or an inaccessible cardinal. Let $\mathbb{P} \in \mathrm{V}_\delta$ be a partial order with the properties (a) and (b) listed in Theorem 1.5. Then there is a $\mathbb{P}$-name $\dot{\mathbb{Q}} \in \mathrm{V}_\delta$ for a partial order such that the statements (i)-(iii) listed in Theorem 1.5 hold whenever $G * H$ is $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V.*

Note that, if $\kappa = \omega_1$ and U witnesses that δ is a measurable cardinal in V, then $(\star)$ holds and the above remarks imply that the well-ordering of $\mathrm{H}(\omega_2)$ produced by the above forcing has the optimal complexity that is compatible with the existence of a measurable cardinal. Since the above construction still relies on the assumption that V lies close to some well-behaved inner model, we may ask if it is possible to have such well-orders in the presence of larger large cardinals. It was pointed out to the authors by Daisuke Ikegami that results of Woodin on the Π_2-maximality of the $\mathbb{P}_{\max}$-extension of $\mathrm{L}(\mathbb{R})$ (see [14] and [20]) directly imply that the existence of certain large cardinals implies that no well-order of $\mathrm{H}(\omega_2)$ is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula with parameter ω_1.

Proposition 1.8. *Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them. If there is a well-order of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula with parameter $z \subseteq \omega_1$, then $z \notin \mathrm{L}(\mathbb{R})$.*

Proof. Given $\alpha < \omega_1$, let WO_α denote the set of all $x \in \mathbb{R}$ that code (in some fixed canonical way) a well-ordering of ω of order-type α. Note that the set $\{\langle x, \alpha\rangle \in \mathbb{R} \times \omega_1 \mid x \in \mathrm{WO}_\alpha\}$ is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula without parameters. Let $\mathcal{L}_{\dot{A}}$ denote the language of set theory extended by an unary predicate symbol $\dot{A}$. Then there is a Σ_1-formula $\varphi_0(v)$ in $\mathcal{L}_{\dot{A}}$ such that the axioms of ZFC prove that

$$\langle \mathrm{H}(\omega_2), \in, A\rangle \models \varphi_0(z) \iff z \subseteq \{\alpha < \omega_1 \mid \exists x \in A \ x \in \mathrm{WO}_\alpha\}$$

for all $A \subseteq \mathbb{R}$ and $z \in \mathrm{H}(\omega_2)$. Moreover, there is a Π_1-formula $\varphi_1(v)$ in $\mathcal{L}_{\dot{A}}$ such that the axioms of ZFC prove that

$$\langle \mathrm{H}(\omega_2), \in, A\rangle \models \varphi_1(z) \iff \{\alpha < \omega_1 \mid \exists x \in A \ x \in \mathrm{WO}_\alpha\} \subseteq z$$

for all $A \subseteq \mathbb{R}$ and $z \in \mathrm{H}(\omega_2)$.

Fix a Σ_1-formula $\psi(v_0, v_1, v_2)$ in the language of set theory. Using the formulas constructed above and the arguments used in the proof of Proposition 1.1, we find a Π_2-sentence Ψ in $\mathcal{L}_{\dot{A}}$ such that the axioms of ZFC prove that whenever $A \subseteq \mathbb{R}$ and $z_A = \{\alpha < \omega_1 \mid \exists x \in A \; x \in \mathrm{WO}_\alpha\}$, then the set

$$\{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} \mid \langle \mathrm{H}(\omega_2), \in\rangle \models \psi(x, y, z_A)\}$$

is a well-ordering of $\mathbb{R}$ if and only if $\langle \mathrm{H}(\omega_2), \in, A\rangle \models \Psi$.

Assume, towards a contradiction, that there are infinitely many Woodin cardinals with a measurable cardinal above them and there is $z \in \mathcal{P}(\omega_1)^{\mathrm{L}(\mathbb{R})}$ such that

$$\{\langle x, y\rangle \in \mathrm{H}(\omega_2) \times \mathrm{H}(\omega_2) \mid \langle \mathrm{H}(\omega_2), \in\rangle \models \psi(x, y, z)\}$$

is a well-ordering of $\mathrm{H}(\omega_2)$. Set $A_z = \bigcup_{\alpha \in z} \mathrm{WO}_\alpha \in \mathrm{L}(\mathbb{R})$. Then $z = z_{A_z}$ and $\langle \mathrm{H}(\omega_2), \in, A_z\rangle \models \Psi$. Let G be $\mathbb{P}_{\max}$-generic over $\mathrm{L}(\mathbb{R})$. By [14, Theorem 7.3], we have $\langle \mathrm{H}(\omega_2)^{\mathrm{L}(\mathbb{R})[G]}, \in, A_z\rangle \models \Psi$ and hence there is a well-order of $\mathbb{R}$ that is definable over $\langle \mathrm{H}(\omega_2)^{\mathrm{L}(\mathbb{R})[G]}, \in\rangle$ by the formula ψ and the parameter $z \in \mathrm{L}(\mathbb{R})$. By our assumptions and [14, Lemma 2.10], the partial order $\mathbb{P}_{\max}$ is homogeneous[5] in $\mathrm{L}(\mathbb{R})$. This shows that the set

$$\{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} \mid \mathbb{1} \Vdash^{\mathrm{L}(\mathbb{R})}_{\mathbb{P}_{\max}} \psi(\check{x}, \check{y}, \check{z})\} \;\in\; \mathrm{L}(\mathbb{R})$$

is a well-order of $\mathbb{R}$ in $\mathrm{L}(\mathbb{R})$. By results of Woodin (see [18, 8.24 Theorem]), our assumptions imply that AD holds in $\mathrm{L}(\mathbb{R})$ and hence $\mathrm{L}(\mathbb{R})$ contains no well-orders of $\mathbb{R}$, a contradiction. □

Since the existence of a well-order of $\mathrm{H}(\omega_2)$ definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula with parameter ω_1 is consistent with the existence of a measurable cardinal and inconsistent with the existence of infinitely many Woodin cardinals with a measurable cardinal above them, it is natural to ask the following question.

Question 1.9. *What is the weakest large cardinal whose existence implies that no well-order of* $\mathrm{H}(\omega_2)$ *is locally definable by a* Σ_1*-formula with parameter* ω_1*?*

This question will be answered in the forthcoming [16], by showing that the existence of a well-ordering of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula with parameter ω_1 is consistent with the existence of a Woodin cardinal and inconsistent with the existence of a Woodin cardinal with a measurable cardinal above it. Moreover, we will use the techniques

[5]In the sense that for all conditions $p_0, p_1 \in \mathbb{P}_{\max}$, there are conditions $q_0, q_1 \in \mathbb{P}_{\max}$ such that $q_i \leq_{\mathbb{P}_{\max}} p_i$ and the restriction of $\mathbb{P}_{\max}$ to all conditions below q_0 is isomorphic to the restriction of $\mathbb{P}_{\max}$ to all conditions below q_1.

developed in Section 7 of this paper to show that the existence of such a well-order is compatible with the existence of a Woodin cardinal and a failure of the GCH at ω_1.

In another direction, the above arguments do not answer the above question for the case $\kappa > \omega_1$.

Question 1.10. *Given a formula $\varphi(v_0, v_1, v_2)$, do the axioms of* ZFC *prove that for every supercompact cardinal δ and every regular cardinal $\omega_1 < \kappa < \delta$, the set*

$$\{\langle x, y\rangle \in \mathrm{H}(\kappa^+) \times \mathrm{H}(\kappa^+) \mid \langle \mathrm{H}(\kappa^+), \in\rangle \models \varphi(x, y, \kappa)\}$$

is not a well-ordering of $\mathrm{H}(\kappa^+)$*?*

Next, we consider locally definable well-orders of the set ${}^\kappa\kappa$ of all functions from κ to κ. Note that the Σ_1-Reflection Principle implies that the set ${}^\kappa\kappa$ is not definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters in $\mathrm{H}(\kappa)$ and hence no well-order of this set is definable in this way. In the case of successor cardinals, the following proposition shows that the forcing construction provided by Theorem 1.5 adds a locally Π_1-definable well-order of ${}^\kappa\kappa$ whose definition uses parameters of small cardinality.

Proposition 1.11. *Assume that η is an infinite cardinal, $\kappa = \eta^+$ and there is a well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameter κ. Then there is a well-order of ${}^\kappa\kappa$ that is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Π_1-formula with parameter η.*

Proof. Fix a Σ_1-formula $\Phi(v_0, v_1, v_2)$ such that the formula Φ and the parameter κ define a well-order $\lhd$ of $\mathrm{H}(\kappa^+)$ over $\langle \mathrm{H}(\kappa^+), \in\rangle$. Let A be the set of all $x \in \mathrm{H}(\kappa^+)$ with $|x| = \kappa$. Then A consists of all $x \in \mathrm{H}(\kappa^+)$ such that there is no surjection from η onto x in $\mathrm{H}(\kappa^+)$. Therefore A is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Π_1-formula $\psi(v_0, v_1)$ with parameter η. If $x \in A$, then κ is the unique ordinal $\alpha < \kappa^+$ such that there is a bijection $b : \alpha \longrightarrow x$ in $\mathrm{H}(\kappa^+)$ and for every $\bar{\alpha} < \alpha$ there is a surjection $s : \eta \longrightarrow \bar{\alpha}$ in $\mathrm{H}(\kappa^+)$. This shows that there is a Σ_1-formula $\varphi(v_0, v_1, v_2)$ such that

$$y = \kappa \iff \langle \mathrm{H}(\kappa^+), \in\rangle \models \varphi(x, y, \eta)$$

holds for all $x, y \in \mathrm{H}(\kappa^+)$ with $x \in A$. Finally, pick a Σ_0-formula $\phi(v_0, v_1)$ such that $\phi(x, y)$ holds if and only if $x : y \longrightarrow y$ is a function.

Define $\blacktriangleleft = \lhd \cap ({}^\kappa\kappa \times {}^\kappa\kappa)$. Then the set $\blacktriangleleft$ is equal to the set of all pairs $\langle x, y\rangle$ in $\mathrm{H}(\kappa^+) \times \mathrm{H}(\kappa^+)$ with

$$\begin{aligned}\langle \mathrm{H}(\kappa^+), \in\rangle \ \models x \neq y \ \wedge \ \psi(x, \eta) \ \wedge \ \psi(y, \eta) \\ \wedge \ \forall \alpha \ [\varphi(x, \alpha, \eta) \ \longrightarrow \ (\neg\Phi(y, x, \alpha) \wedge \phi(x, \alpha) \wedge \phi(y, \alpha)]\end{aligned}$$

This shows that the well-ordering $\blacktriangleleft$ of ${}^{\kappa}\kappa$ is definable over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Π_1-formula with parameter η. □

Since the set $\{\omega\}$ is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_0-formula without parameters, the above proposition shows that the forcing given by Theorem 1.5 produces a locally definable well-ordering of ${}^{\omega_1}\omega_1$ of optimal complexity.

Corollary 1.12. *Assume that there is a well-order of* $\mathrm{H}(\omega_2)$ *that is definable over* $\langle \mathrm{H}(\omega_2), \in\rangle$ *by a* Σ_1*-formula with parameter* ω_1*. Then there is a well-order of* ${}^{\omega_1}\omega_1$ *that is definable over* $\langle \mathrm{H}(\omega_2), \in\rangle$ *by a* Π_1*-formula without parameters.* □

Finally, motivated by [11, Corollary 1.5], we also consider *Bernstein subsets of* ${}^{\kappa}\kappa$. Given an uncountable regular cardinal κ, we equip ${}^{\kappa}\kappa$ with the topology whose basic open subsets are of the form $N_s = \{x \in {}^{\kappa}\kappa \mid s \subseteq x\}$ for some function $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$ and we say that a closed subset of this space is *perfect* if it is homeomorphic to the set ${}^{\kappa}2$ equipped with the subspace topology induced by that of ${}^{\kappa}\kappa$. A *Bernstein subset of* ${}^{\kappa}\kappa$ is a subset X of ${}^{\kappa}\kappa$ with the property that X and its complement intersect every perfect subset of ${}^{\kappa}\kappa$.

By [11, Corollary 1.5], the partial order $\mathbb{P}$ that witnesses Theorem 1.3 introduces a Bernstein subset of ${}^{\kappa}\kappa$ that is Δ_1-definable with parameters over $\langle \mathrm{H}(\kappa^+), \in\rangle$. It is easy to see that the proof of this result, presented in [11, Section 4], also shows that in the forcing extensions produced by the above theorems there are Bernstein subsets of ${}^{\kappa}\kappa$ with simple definitions.

Corollary 1.13.
(i) In the situation of Theorem 1.4, forcing with the partial order $\mathbb{P}$ *introduces a Bernstein subset of* ${}^{\kappa}\kappa$ *that is* Δ_1*-definable over* $\langle \mathrm{H}(\kappa^+), \in\rangle$ *using the parameter* $\langle S_\alpha \mid \alpha < \kappa\rangle$ *.*
(ii) In the situation of Theorem 1.5 or Theorem 1.7, forcing with the partial order $\mathbb{P} * \dot{\mathbb{Q}}$ *introduces a Bernstein subset of* ${}^{\kappa}\kappa$ *that is* Δ_1*-definable over* $\langle \mathrm{H}(\kappa^+), \in\rangle$ *using the parameter* κ*.* □

We outline the structure of this paper. In Section 2, we start by discussing forcing techniques that allow us to make an arbitrary subset of $\mathrm{H}(\kappa^+)$ definable in a generic extension of the ground model. Section 3 contains the definition of *strongly* S*-complete forcings* and several observations that will later allow us to show that the forcing constructed in the proof of Theorem 1.4 is $<\kappa$-distributive and preserves the stationarity of certain subsets of κ. Next, we prove that the generic coding introduced in Section 2 is absolute with respect to strongly S-complete forcings. We continue by constructing the partial order witnessing Theorem 1.4 in Section 5 and proving the statements of the theorem in Section 6. Finally, we prove Theorem 1.5 and Theorem 1.7 in Section 7 by constructing forcing extensions of canonical inner models that contain simply definable sequences of disjoint fat stationary sets.

2. Almost Disjoint Coding at Uncountable Cardinals

In this section, we discuss *almost disjoint coding forcing* (see [3] and [12]) for uncountable cardinals κ that satisfy $\kappa = \kappa^{<\kappa}$. Given such κ, this forcing technique will allow us to make an arbitrary subset of ${}^\kappa\kappa$ definable by a formula of low complexity in an upwards-absolute way. In order to make this precise, we generalize basic notions of complexity to our uncountable setting.

We equip the set ${}^\kappa\kappa$ with the topology whose basic open sets are of the form $N_s = \{x \in {}^\kappa\kappa \mid s \subseteq x\}$ for some function $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$. A subset of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^0_2$*-subset* of ${}^\kappa\kappa$ if it is equal to the union of κ-many closed subsets of ${}^\kappa\kappa$. Note that a subset of ${}^\kappa\kappa$ is closed if and only if it is equal to the set

$$[T] \ = \ \{x \in {}^\kappa\kappa \mid \forall\alpha < \kappa \ x \restriction \alpha \in T\}$$

of κ-branches through some subtree[6] T of ${}^{<\kappa}\kappa$. In particular, $A \subseteq {}^\kappa\kappa$ is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$ if and only if there is a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ of subtrees of ${}^{<\kappa}\kappa$ with $A = \bigcup_{\alpha<\kappa}[T_\alpha]$. We say that such a sequence of trees *witnesses that A is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$.*

We will now discuss how *almost disjoint coding forcing* at uncountable cardinals κ with $\kappa = \kappa^{<\kappa}$ allows us to make an arbitrary subset of ${}^\kappa\kappa$ $\mathbf{\Sigma}^0_2$-definable by a cofinality-preserving forcing.

Definition 2.1. Assume that κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $A \subseteq {}^\kappa\kappa$ and $\vec{s} = \langle s_\alpha \mid \alpha < \kappa\rangle$ is an enumeration of ${}^{<\kappa}\kappa$ with the property that every element of ${}^{<\kappa}\kappa$ is enumerated κ-many times. We define a partial order $\mathbb{C}_{\vec{s}}(A)$ by the following clauses.

(i) A condition in $\mathbb{C}_{\vec{s}}(A)$ is a pair $p = \langle t_p, a_p\rangle$ with $t_p : \alpha_p \longrightarrow 2$ for some $\alpha_p < \kappa$ and $a_p \in [A]^{<\kappa}$.

(ii) We have $q \leq_{\mathbb{C}_{\vec{s}}(A)} p$ if and only if $t_p \subseteq t_q$, $a_p \subseteq a_q$ and

$$s_\beta \subseteq x \ \longrightarrow \ t_q(\beta) = 1$$

for every $x \in a_p$ and $\alpha_p \leq \beta < \alpha_q$.

The following proposition lists the basic properties of this partial order.

[6] A nonempty subset $T \subseteq {}^{<\kappa}\kappa$ is a *subtree of* ${}^{<\kappa}\kappa$ if it is closed under initial segments.

Proposition 2.2. *(i) If $\eta < \kappa$ and $\langle p_\xi \mid \xi < \eta\rangle$ is a descending sequence of conditions in $\mathbb{C}_{\vec{s}}(A)$, then the pair $p = \langle \bigcup_{\xi<\eta} t_{p_\xi}, \bigcup_{\xi<\eta} a_{p_\xi}\rangle$ is a condition in $\mathbb{C}_{\vec{s}}(A)$ with $p \leq_{\mathbb{C}_{\vec{s}}(A)} p_\xi$ for all $\xi < \eta$. In particular, $\mathbb{C}_{\vec{s}}(A)$ is $<\kappa$-closed with infima.*

(ii) If p and q are conditions in $\mathbb{C}_{\vec{s}}(A)$ with $t_p = t_q$, then p and q are compatible in $\mathbb{C}_{\vec{s}}(A)$. In particular, $\mathbb{C}_{\vec{s}}(A)$ satisfies the κ^+-chain condition. □

The coding provided by the above forcing turns out to be much stronger than in the countable setting, because we no longer need to bother with the definability of ${}^\kappa\kappa$ of the ground model (see [9, Section 1]).

There is a sequence $\langle \dot{T}_\alpha \mid \alpha < \kappa\rangle$ of canonical $\mathbb{C}_{\vec{s}}(A)$-names with the property that whenever G is $\mathbb{C}_{\vec{s}}(A)$-generic over V and $t_G = \bigcup\{t_p \mid p \in G\} : \kappa \longrightarrow 2$, then for every $\alpha < \kappa$,

$$\dot{T}_\alpha^G = \{t \in {}^{<\kappa}\kappa \mid \forall \alpha < \beta < \kappa \ [t_G(\beta) = 0 \longrightarrow s_\beta \nsubseteq t]\}.$$

Theorem 2.3. *If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, A is a subset of ${}^\kappa\kappa$ and G is $\mathbb{C}_{\vec{s}}(A)$-generic over* V*, then the sequence $\langle \dot{T}_\alpha^G \mid \alpha < \kappa\rangle$ witnesses that A is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$ in* V$[G]$.

Rather than presenting the short proof of this theorem, we will prove an *absoluteness version* of it in Section 4 (see Corollary 4.3) that will imply the above statement. This result will show that the above coding does not only hold true in V$[G]$ but is in fact persistent under certain further forcing.

We close this section with a small observation showing that in a certain sense, the generic coding provided by Theorem 2.3 is optimal.

Proposition 2.4. *Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$. Then there is a subset C of ${}^\kappa\kappa$ with the property that in every generic extension of the ground model by a $<\kappa$-distributive forcing, the complement of C is not a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$.*

Proof. Define $\mathcal{C}$ to be the *club filter* on κ, i.e. $\mathcal{C}$ is the set of all $x \in {}^\kappa\kappa$ such that the set $\{\alpha < \kappa \mid x(\alpha) = 0\}$ contains a closed unbounded subset of κ. Assume, towards a contradiction, that there is a generic extension V$[G]$ of the ground model V by a $<\kappa$-distributive forcing $\mathbb{P}$ such the complement of $\mathcal{C}$ is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$ in V$[G]$. Work in V$[G]$. By our assumption, there is a sequence $\langle U_\alpha \mid \alpha < \kappa\rangle$ of open subsets of ${}^\kappa\kappa$ with $\mathcal{C} = \bigcap_{\alpha<\kappa} U_\alpha$. By the $<\kappa$-distributivity of $\mathbb{P}$, the set $\mathcal{C}$ is dense in ${}^\kappa\kappa$ and this implies that the set U_α is open dense for every $\alpha < \kappa$.

Let $h : {}^\kappa\kappa \longrightarrow {}^\kappa\kappa$ be the unique function such that $x(\alpha) \geq 2$ implies $h(x)(\alpha) = x(\alpha)$ and $x(\alpha) < 2$ implies $h(x)(\alpha) = 1 - x(\alpha)$ for all $x \in {}^\kappa\kappa$ and $\alpha < \kappa$. Then h is a homeomorphism of ${}^\kappa\kappa$ and $h[\mathcal{C}]$ is also equal to the interseaction of κ-many dense open subsets of ${}^\kappa\kappa$. In this situation, a standard argument shows that there is an $x \in \mathcal{C} \cap h[\mathcal{C}]$ and there are closed unbounded subsets C_0 and C_1 of κ in V such that $C_i \subseteq \{\alpha < \kappa \mid x(\alpha) = i\}$ for all $i < 2$. But then $C_0 \cap C_1 = \emptyset$, a contradiction. □

Note that it also possible to use [8, Theorem 4.2] to derive a contradiction in the proof of the above proposition.

3. Strongly S-complete Forcings

In this section, we define a strengthening of the notion of an *S-complete forcing* introduced in [19, Chapter V, Definition 1.1]. We will use this property to show that the forcing constructed in the proof of Theorem 1.4 is $<\kappa$-distributive and preserves the stationarity of certain subsets of κ. This will allow us to generically code information with the help of the sequence $\langle S_\alpha \mid \alpha \leq \kappa\rangle$ of disjoint stationary sets.

Definition 3.1. Assume κ is an uncountable regular cardinal, $S \subseteq \kappa$ and $\mathbb{P}$ is a partial order. We say that $\mathbb{P}$ is *strongly S-complete* if there is a sequence $\vec{D} = \langle D_\alpha \mid \alpha < \kappa\rangle$ of open dense subsets of $\mathbb{P}$ with the property that whenever

(a) $\theta > \kappa$ is a regular cardinal with $\mathcal{P}(\mathbb{P}) \in \mathrm{H}(\theta)$,
(b) M is an elementary submodel of $\mathrm{H}(\theta)$ of cardinality less than κ such that $\vec{D}, \mathbb{P} \in M$ and $\alpha = \kappa \cap M \in S$, and
(c) $\vec{p} = \langle p_\xi \mid \xi < \eta\rangle$ is a descending sequence of conditions in $\mathbb{P}$ such that $p_\xi \in M$ for every $\xi < \eta$ and $\{\bar{\alpha} < \alpha \mid \exists \xi < \eta \; p_\xi \in D_{\bar{\alpha}}\}$ is unbounded in α,

then there is a condition p in $\mathbb{P}$ with $p \leq_{\mathbb{P}} p_\xi$ for every $\xi < \eta$.

Lemma 3.2. *Let κ be an uncountable regular cardinal with $\eta^{<\eta} < \kappa$ for every cardinal $\eta < \kappa$, let S be a fat stationary subset of κ and let $\mathbb{P}$ be a strongly S-complete partial order. Then $\mathbb{P}$ is $<\kappa$-distributive.*

Proof. We show that $\mathbb{P}$ is $<\eta$-distributive for every infinite cardinal $\eta \leq \kappa$ by induction on η. Note that limit steps are trivial, for if η is a limit cardinal, then $\mathbb{P}$ is $<\eta$-distributive iff it is $<\mu$-distributive for every infinite cardinal $\mu < \eta$. Moreover, any forcing is trivially $<\omega$-distributive. Similarly, if $\eta < \kappa$

is a singular cardinal such that $\mathbb{P}$ is $<\eta$-distributive, then $\mathbb{P}$ is in fact $<\eta^+$-distributive. All of the above hold true for any notion of forcing P. The only nontrivial case in the induction is the following.

Assume $\eta < \kappa$ is a regular cardinal such that $\mathbb{P}$ is $<\eta$-distributive, let $\dot{f}$ be a $\mathbb{P}$-name for a function from η to On and let p_0 be a condition in $\mathbb{P}$. Pick a sequence $\vec{D} = \langle D_\alpha \mid \alpha < \kappa\rangle$ of open dense subsets of $\mathbb{P}$ witnessing that $\mathbb{P}$ is strongly S-complete, and pick a sufficiently large regular cardinal θ.

We inductively construct a continuous $\subseteq$-increasing chain $\langle M_\beta \mid \beta < \kappa\rangle$ of elementary submodels of $\mathrm{H}(\theta)$ of cardinality less than κ and a strictly increasing continuous sequence $\langle \alpha_\beta \mid \beta < \kappa\rangle$ of limit ordinals less than κ such that the following statements hold for every $\beta < \kappa$.

(i) $\eta, \dot{f}, p_0, \vec{D}, \mathbb{P} \in M_0$.
(ii) $\alpha_\beta = \kappa \cap M_\beta \in \kappa$.
(iii) If $\mathrm{cof}(\alpha_\beta) < \eta$, then ${}^{\mathrm{cof}(\alpha_\beta)}M_\beta \subseteq M_{\beta+1}$.

Then $C = \{\alpha_\beta \mid \beta < \kappa\}$ is a closed unbounded subset of κ and, using the assumption that S is fat stationary, there is a strictly increasing continuous map $b : \eta + 1 \longrightarrow \kappa$ such that $\{\alpha_{b(\xi)} \mid \xi \leq \eta\}$ is a closed subset of S of order-type $\eta+1$. Now we inductively construct a sequence $\langle p_\xi \mid \xi < \eta\rangle$ of conditions in $\mathbb{P}$ and a sequence $\langle t_\xi \mid \xi < \eta\rangle$ such that $p_\xi \in M_{b(\xi+1)}$, $p_{\xi+1} \Vdash$ "$\dot{f} \restriction \check{\xi} = \check{t}_\xi$" and $p_{\xi+1} \in D_{\alpha_{b(\xi)}}$ for every $\xi < \eta$.

Assume $\langle p_{\bar{\xi}} \mid \bar{\xi} \leq \xi\rangle$ is already constructed. Then p_ξ and $D_{\alpha_{b(\xi)}}$ are both elements of $M_{b(\xi+1)}$, because $\alpha_{b(\xi)} < \alpha_{b(\xi+1)} \subseteq M_{b(\xi+1)}$. Using that $\mathbb{P}$ is $<\eta$-distributive by induction hypothesis, there are $p_{\xi+1} \in D_{\alpha_{b(\xi)}} \cap M_{b(\xi+1)+1} \subseteq M_{b(\xi+2)}$ and t_ξ with $p_{\xi+1} \leq_{\mathbb{P}} p_\xi$ and $p_{\xi+1} \Vdash$ "$\dot{f} \restriction \check{\xi} = \check{t}_\xi$".

Now, assume $\xi \in \eta \cap \mathrm{Lim}$ and $\langle p_{\bar{\xi}} \mid \bar{\xi} < \xi\rangle$ is already constructed. Then $\{\bar{\alpha} < \alpha_{b(\xi)} \mid \exists \bar{\xi} < \xi \ p_{\bar{\xi}} \in D_{\bar{\alpha}}\}$ is unbounded in $\alpha_{b(\xi)}$. Hence there is a condition q in $\mathbb{P}$ with $q \leq_{\mathbb{P}} p_{\bar{\xi}}$ for every $\bar{\xi} < \xi$. Let $\langle \xi_\mu \mid \mu < \mathrm{cof}(\xi)\rangle$ be a strictly increasing continuous sequence of ordinals that is cofinal in ξ. Since $\mathrm{cof}(\alpha_{b(\xi)}) = \mathrm{cof}(\xi) < \eta$, we can conclude ${}^{\mathrm{cof}(\xi)}M_{b(\xi)} \subseteq M_{b(\xi)+1} \subseteq M_{b(\xi+1)}$. This shows that the sequence $\langle p_{\xi_\mu} \mid \mu < \mathrm{cof}(\xi)\rangle$ is an element of $M_{b(\xi+1)}$ and, by elementarity, there is a condition $p_\xi \in M_{b(\xi+1)}$ such that $p_\xi \leq_{\mathbb{P}} p_{\bar{\xi}}$ holds for every $\bar{\xi} < \xi$.

Since $\{\bar{\alpha} < \alpha_{b(\eta)} \mid \exists \xi < \eta \ p_\xi \in D_{\bar{\alpha}}\}$ is unbounded in $\alpha_{b(\eta)}$, there is a condition $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} p_\xi$ for every $\xi < \eta$. If we define $f = \bigcup_{\xi<\eta} t_\xi$, then $p \Vdash$ "$\dot{f} = \check{f}$". This shows that $\mathbb{P}$ is η-distributive. □

We continue with a technical definition that summarizes the important properties that we will deduce from strong S-completeness.

Definition 3.3. Let $\mathbb{P}$ be a partial order, let θ be a regular uncountable cardinal with $\mathcal{P}(\mathbb{P}) \in \mathrm{H}(\theta)$ and let M be an elementary submodel of $\mathrm{H}(\theta)$ with $\mathbb{P} \in M$. A condition p in $\mathbb{P}$ is *strongly* $(M, \mathbb{P})$*-generic* if whenever D is a open dense subset of $\mathbb{P}$ that is an element of M, then there is a condition $q \in D$ with $q \in M$ and $p \leq_{\mathbb{P}} q$.

The following basic observation will be needed in the proof of the next lemma.

Proposition 3.4. *Let κ be an uncountable regular cardinal and let S be a fat stationary subset of κ. If $\mathbb{P}$ is a $<\kappa$-closed partial order, then S is a fat stationary subset of κ in every $\mathbb{P}$-generic extension of* V.

Proof. Let $\dot{C}$ be a name for a club subset of κ, $\gamma < \kappa$ and p_0 be a condition in $\mathbb{P}$. Then we can inductively construct a descending sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ of conditions in $\mathbb{P}$ and a sequence $\langle c_\alpha \mid \alpha < \kappa \rangle$ of bounded subsets of κ such that $p_{\alpha+1} \Vdash$ "$\dot{C} \cap \check{\alpha} = \check{c}_\alpha$" holds for every $\alpha < \kappa$. Then $C = \bigcup_{\alpha<\kappa} c_\alpha$ is a closed unbounded subset of κ and there is an $\alpha_* < \kappa$ such that $C \cap S \cap \alpha_*$ contains a closed subset of order-type γ. Hence $p_{\alpha_*} \leq_{\mathbb{P}} p_0$ forces that the intersection of $\dot{C}$ and S contains a closed subset of order-type γ. □

We are now ready to show that strongly S-complete forcings contain dense subsets of strongly generic conditions for a great variety of elementary submodels.

Lemma 3.5. *Assume that*

(a) *κ is an uncountable regular cardinal with $\eta^{<\eta} < \kappa$ for every $\eta < \kappa$,*
(b) *S is a fat stationary subset of κ,*
(c) *$\mathbb{P}$ is a partial order that is $<\kappa$-closed with infima,*
(d) *$\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a strongly S-complete partial order,*
(e) *$\vec{D} = \langle \dot{D}_\alpha \mid \alpha < \kappa \rangle$ is a sequence of $\mathbb{P}$-names for open dense subsets of $\dot{\mathbb{Q}}$ such that whenever G is $\mathbb{P}$-generic over* V, *then the sequence $\langle \dot{D}_\alpha^G \mid \alpha < \kappa \rangle$ witnesses that $\dot{\mathbb{Q}}^G$ is strongly S-complete in* $\mathrm{V}[G]$, *and*
(f) *$\theta > \kappa$ is regular with $\vec{D}, \mathcal{P}(\mathbb{P} * \dot{\mathbb{Q}}) \in \mathrm{H}(\theta)$.*

Then for every element x of $\mathrm{H}(\theta)$ and every regular cardinal $\eta < \kappa$, there is a dense set of conditions $\langle p, \dot{q} \rangle$ in $\mathbb{P} * \dot{\mathbb{Q}}$ such that the following statements hold for some elementary submodel M of $\mathrm{H}(\theta)$.

(i) *M has cardinality less than κ and $x, \mathbb{P}, \dot{\mathbb{Q}} \in M$.*

(ii) $\alpha = \kappa \cap M \in S$ *and there is a closed unbounded subset of* α *of order-type* η *consisting of elements of* S*.*

(iii) $\langle p, \dot{q} \rangle$ *is strongly* $(M, \mathbb{P} * \dot{\mathbb{Q}})$*-generic.*

(iv) The condition p *is the infimum of a descending sequence of conditions in* $\mathbb{P} \cap M$*.*

Proof. We start by proving some general facts about the behaviour of elementary submodels of $\mathrm{H}(\theta)$ in $\mathbb{P}$-generic extensions. Let M be an elementary submodel of $\mathrm{H}(\theta)$ of size less than κ, with $\vec{D}, \mathbb{P}, \dot{\mathbb{Q}} \in M$ and $\kappa \cap M \in S$, let $\langle \langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}} \rangle \mid \bar{\xi} < \xi \rangle$ be a descending sequence of conditions in $(\mathbb{P} * \dot{\mathbb{Q}}) \cap M$ of length less than κ, such that every dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$ that is an element of M has some $p_{\bar{\xi}}$ as element, and let p_ξ be the infimum of the sequence $\langle p_{\bar{\xi}} \mid \bar{\xi} < \xi \rangle$ in $\mathbb{P}$. If G is $\mathbb{P}$-generic over V, then $\mathrm{H}(\theta)^{\mathrm{V}[G]}$ is a $\mathbb{P}$-generic extension of $\mathrm{H}(\theta)^{\mathrm{V}}$, and we define

$$M[G] \; = \; \{\dot{x}^G \mid \dot{x} \in M \text{ is a } \mathbb{P}\text{-name}\}.$$

The following claims are standard, but we provide their short proofs for sake of completeness.

Claim. *If* G *is* $\mathbb{P}$*-generic over* V *with* $p_\xi \in G$*, then* $M[G]$ *is an elementary submodel of* $\mathrm{H}(\theta)^{\mathrm{V}[G]}$ *with* $M \cap \kappa = M[G] \cap \kappa$*.*

Proof of the Claim. Let $\dot{x} \in M$ be a $\mathbb{P}$-name with $\dot{x}^G \in \kappa$. Then

$$D \; = \; \{\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \mid \exists \alpha < \kappa \; p \Vdash \text{“}\dot{x} = \check{\alpha}\text{”}\}$$

is a open dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$ an an element M. Hence $\langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}} \rangle \in D \cap M$ for some $\bar{\xi} < \xi$ and there is an $\bar{\alpha} \in \kappa \cap M$ with $\dot{x}^G = \bar{\alpha}$.

Assume there are $\mathbb{P}$-names $\dot{x}_0, \ldots, \dot{x}_{n-1} \in M$ such that $\varphi(x, \dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ holds in $\mathrm{H}(\theta)^{\mathrm{V}[G]}$ for some formula $\varphi(v_0, \ldots, v_n)$ and some $x \in \mathrm{H}(\theta)^{\mathrm{V}[G]}$. Since $\mathrm{H}(\theta)^{\mathrm{V}[G]}$ is a $\mathbb{P}$-generic extension of the model $\mathrm{H}(\theta)^{\mathrm{V}}$, the set of all conditions $\langle p, \dot{q} \rangle$ in $\mathbb{P} * \dot{\mathbb{Q}}$ such that either $p \Vdash$ “$\forall x \, \neg\varphi(x, \dot{x}_0, \ldots, \dot{x}_{n-1})$” holds in $\mathrm{H}(\theta)^{\mathrm{V}}$ or $p \Vdash \varphi(\dot{x}, \dot{x}_0, \ldots, \dot{x}_{n-1})$ holds in $\mathrm{H}(\theta)^{\mathrm{V}}$ for some $\mathbb{P}$-name $\dot{x} \in \mathrm{H}(\theta)^{\mathrm{V}}$ is a open dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$, and is an element of M. By our assumptions, this implies that there is a $\mathbb{P}$-name $\dot{x}_* \in M$ and a $\bar{\xi} < \xi$ such that $p_{\bar{\xi}} \Vdash \varphi(\dot{x}_*, \dot{x}_0, \ldots, \dot{x}_{n-1})$ holds in $\mathrm{H}(\theta)^{\mathrm{V}}$ and this shows that there is an $x \in M[G]$ such that $\varphi(x, \dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ holds in $\mathrm{H}(\theta)^{\mathrm{V}[G]}$. □

Claim. *There is a* $\mathbb{P}$*-name* $\dot{q}_\xi$ *for a condition in* $\dot{\mathbb{Q}}$ *such that* $\langle p_\xi, \dot{q}_\xi \rangle$ *is strongly* $(M, \mathbb{P} * \dot{\mathbb{Q}})$*-generic condition and* $\langle p_\xi, \dot{q}_\xi \rangle \leq_{\mathbb{P} * \dot{\mathbb{Q}}} \langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}} \rangle$ *for every* $\bar{\xi} < \xi$*.*

Proof of the Claim. Let G be $\mathbb{P}$-generic over V with $p_\xi \in G$ and let D be a open dense subset of $\dot{\mathbb{Q}}^G$ that is an element of $M[G]$. Then there is a $\mathbb{P}$-name $\dot{D} \in M$ for a dense subset of $\dot{\mathbb{Q}}$ with $D = \dot{D}^G$ and

$$D_* \ = \ \{\langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}} \mid p \Vdash \text{“}\dot{q} \in \dot{D}\text{”}\}$$

is a open dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$ and an element of M. Hence there is a $\bar{\xi} < \xi$ such that $\langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}}\rangle$ is an element of $D_* \cap M$, and this yields $\dot{q}_{\bar{\xi}}^G \in D$. We can conclude that for every $\alpha \in \kappa \cap M[G]$ there is $\bar{\xi} < \xi$ with $\dot{q}_{\bar{\xi}}^G \in \dot{D}_\alpha^G$. By our assumptions and the above claim, this shows that there is a condition q_ξ in $\dot{\mathbb{Q}}^G$ such that $q_\xi \leq_{\dot{\mathbb{Q}}^G} \dot{q}_{\bar{\xi}}^G$ holds in V$[G]$ for every $\bar{\xi} < \xi$. These computations yield a $\mathbb{P}$-name $\dot{q}_\xi$ with the desired properties. □

Pick an element x of H(θ), a condition $\langle p_0, \dot{q}_0\rangle$ in $\mathbb{P} * \dot{\mathbb{Q}}$ and a regular cardinal $\eta < \kappa$. We inductively construct a continuous $\subseteq$-increasing chain M_β of elementary submodels of H(θ) of cardinality less than κ and a strictly increasing continuous sequence $\langle \alpha_\beta \mid \beta < \kappa\rangle$ of limit ordinals less than κ such that the following statements hold for every $\beta < \kappa$.

(i) $\eta, p_0, \dot{q}_0, \vec{D}, \mathbb{P}, \dot{\mathbb{Q}} \in M_0$.
(ii) $\alpha_\beta = \kappa \cap M_\beta \in \kappa$.
(iii) $M_\beta \in M_{\beta+1}$.
(iv) If $\mathrm{cof}(\alpha_\beta) < \eta$, then ${}^{\mathrm{cof}(\alpha_\beta)}M_\beta \subseteq M_{\beta+1}$.

If we define $C = \{\alpha_\beta \mid \beta < \kappa\}$, then there is a continuous strictly increasing function $b : \eta + 1 \longrightarrow \kappa$ such that $\{\alpha_{b(\xi)} \mid \xi \leq \eta\}$ is a closed subset of S of order-type $\eta+1$. We define a decreasing sequence $\langle\langle p_\xi, \dot{q}_\xi\rangle \mid \xi < \eta\rangle$ of conditions in $\mathbb{P}*\dot{\mathbb{Q}}$ such that $\langle p_\xi, \dot{q}_\xi\rangle \in M_{b(\xi+1)}$ and $\langle p_{\xi+1}, \dot{q}_{\xi+1}\rangle$ is an element of every open dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$ that is an element of $M_{b(\xi)}$.

Assume that the sequence $\langle\langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}}\rangle \mid \bar{\xi} \leq \xi\rangle$ is already constructed. Then Lemma 3.2 and Proposition 3.4 show that the partial order $\mathbb{P} * \dot{\mathbb{Q}}$ is $<\kappa$-distributive and this implies that the intersection of all open dense subsets of $\mathbb{P}*\dot{\mathbb{Q}}$ that are elements of $M_{b(\xi)}$ is open dense. Since $p_\xi, \dot{q}_\xi, M_{b(\xi)} \in M_{b(\xi+1)}$ and $M_{b(\xi)}$ has cardinality less than κ in $M_{b(\xi+1)}$, elementarity implies that there is a condition $\langle p_{\xi+1}, \dot{q}_{\xi+1}\rangle$ in $(\mathbb{P} * \dot{\mathbb{Q}}) \cap M_{b(\xi+1)}$ below $\langle p_\xi, \dot{q}_\xi\rangle$ that is an element of every open dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$ that is an element of $M_{b(\xi)}$.

Now assume that $\xi \in \eta \cap \mathrm{Lim}$ and the sequence $\langle\langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}}\rangle \mid \bar{\xi} < \xi\rangle$ is already constructed. By the above claim, there is a strongly $(M_{b(\xi)}, \mathbb{P} * \dot{\mathbb{Q}})$-generic condition r with $r \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}}\rangle$ for every $\bar{\xi} < \xi$. Pick a sequence $\langle \xi_\mu \mid \mu < \mathrm{cof}(\xi)\rangle$ that is cofinal in ξ. Since $\mathrm{cof}(\alpha_{b(\xi)}) = \mathrm{cof}(\xi) < \eta$, our assumptions imply that the sequence $\langle\langle p_{\xi_\mu}, \dot{q}_{\xi_\mu}\rangle \mid \mu < \mathrm{cof}(\xi)\rangle$ is an element $M_{b(\xi+1)}$ and, since $M_{b(\xi)}$ is an element of $M_{b(\xi+1)}$, elementarity implies that there is a strongly $(M_{b(\xi)}, \mathbb{P}*\dot{\mathbb{Q}})$-generic condition $\langle p_\xi, \dot{q}_\xi\rangle \in (\mathbb{P}*\dot{\mathbb{Q}})\cap M_{b(\xi+1)}$ such that $\langle p_\xi, \dot{q}_\xi\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_{\bar{\xi}}, \dot{q}_{\bar{\xi}}\rangle$ holds for every $\bar{\xi} < \xi$.

This completes the construction of the sequence $\langle\langle p_\xi, \dot{q}_\xi\rangle \mid \xi < \eta\rangle$. Another application of the above claim shows that there is a strongly $(M_{b(\eta)}, \mathbb{P}*\dot{\mathbb{Q}})$-generic condition $\langle p_\eta, \dot{q}_\eta\rangle$ such that $\langle p_\eta, \dot{q}_\eta\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_\xi, \dot{q}_\xi\rangle$ for every $\xi < \eta$ and p_η is the infimum of the sequence $\langle p_\xi \mid \xi < \eta\rangle$ in $\mathbb{P}$. Moreover, the set $\{\alpha_{b(\xi)} \mid \xi < \eta\}$ is a club subset of $\alpha_{b(\eta)} \in S$ consisting of elements of S. □

Corollary 3.6. *Assume that κ, S, $\mathbb{P}$, $\dot{\mathbb{Q}}$, $\vec{D}$ and θ satisfy the properties listed in Lemma 3.5. Then S is a fat stationary subset of κ in every $(\mathbb{P}*\dot{\mathbb{Q}})$-generic extension of the ground model.*

Proof. Let $\dot{C}$ be a $(\mathbb{P}*\dot{\mathbb{Q}})$-name for a closed unbounded subset of κ and let $G*H$ be $(\mathbb{P}*\dot{\mathbb{Q}})$-generic over V. Fix an infinite regular cardinal $\eta < \kappa$ such that either η is uncountable or $\kappa = \omega_1$. By Lemma 3.5, there is an elementary submodel M of $\mathrm{H}(\theta)$ containing $\dot{C}$ with $\alpha = \kappa\cap M \in S$, a closed unbounded subset c of α of order-type η contained in S and a strongly $(M, \mathbb{P}*\dot{\mathbb{Q}})$-generic condition $r \in G$. We then have $\alpha \in \mathrm{Lim}(\dot{C}^{G*H})\cap S$ and, if η is uncountable, then $c \cap \dot{C}^{G*H}$ contains a closed subset of order-type η.

If $\kappa > \omega_1$, then this argument shows that $\dot{C}^{G*H} \cap S$ contains a closed subset of order-type $\eta+1$ for every regular cardinal $\eta < \kappa$ and, by [1, Lemma 1.2], this implies that S is fat stationary in $\mathrm{V}[G, H]$. In the other case, the argument shows that S is a stationary subset of ω_1 in $\mathrm{V}[G, H]$ and every such subset is fat by [5]. □

4. Almost Disjoint Coding and strongly S-complete Forcings

This section contains the proofs of the absoluteness versions of Theorem 2.3 mentioned at the end of Section 2. Throughout this section, let κ be an uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$, let $A \subseteq {}^\kappa\kappa$ and let $\langle \dot{T}_\alpha \mid \alpha < \kappa\rangle$ be defined as before the statement of Theorem 2.3.

Lemma 4.1. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{C}_{\vec{s}}(A)$-name for a partial order. Assume that for sufficiently large regular cardinals θ and every $x \in \mathrm{H}(\theta)$ there is a dense set of conditions $\langle p, \dot{q}\rangle$ in $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$ that are strongly $(M, \mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic for some elementary submodel M of $\mathrm{H}(\theta)$ of cardinality less than κ with $x, \mathbb{C}_{\vec{s}}(A), \dot{\mathbb{Q}} \in M$, $\kappa \cap M \in \kappa$ and $a_p \subseteq M$. Then the sequence $\langle \dot{T}^G_\alpha \mid \alpha < \kappa\rangle$ witnesses that A is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G, H]$ whenever $G*H$ is $(\mathbb{C}_{\vec{s}}(A)*\dot{\mathbb{Q}})$-generic over* V, *i.e.* $A = \bigcup_{\alpha<\kappa}[\dot{T}^G_\alpha]$ *in* $\mathrm{V}[G, H]$.

Proof. Fix $x \in A$. Then the set $D = \{p \in \mathbb{C}_{\vec{s}}(A) \mid x \in a_p\}$ is dense and there is a condition $p \in G$ with $x \in a_p$. Assume, towards a contradiction, that $x \notin [\dot{T}^G_{\alpha_p}]^{\mathrm{V}[G]}$. Then there is a $\alpha_p < \beta < \kappa$ with $t_G(\beta) = 0$ and $s_\beta \subseteq x$. In this situation, we can find a $q \in G$ with $q \leq_{\mathbb{C}_{\vec{s}}(A)} p$ and $\beta < \alpha_q$. Then $s_\beta \subseteq x$ implies $t_G(\beta) = t_q(\beta) = 1$, a contradiction. This argument shows that $x \in [\dot{T}^G_{\alpha_p}]^{\mathrm{V}[G]} \subseteq [\dot{T}^G_{\alpha_p}]^{\mathrm{V}[G,H]}$.

Now, assume that there is an $x \in [\dot{T}^G_\alpha]^{\mathrm{V}[G,H]} \setminus A$ for some $\alpha < \kappa$. Pick a $(\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-name $\dot{x}$ for an element of ${}^\kappa\kappa$ with $x = \dot{x}^{G*H}$ and a sufficiently large regular cardinal θ. By our assumption, we can find a condition $\langle p_*, \dot{q}_* \rangle$ in $G * H$ with the property that $\langle p_*, \dot{q}_* \rangle \Vdash$ "$\dot{x} \in [\dot{T}_\alpha] \setminus \check{A}$" and $\langle p_*, \dot{q}_* \rangle$ is strongly $(M, \mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic for some elementary submodel M of $\mathrm{H}(\theta)$ of cardinality less than κ with $\alpha, \dot{x}, \mathbb{C}_{\vec{s}}(A), \dot{\mathbb{Q}} \in M$, $\kappa \cap M \in \kappa$ and $a_{p_*} \subseteq M$.

Given $\bar{\alpha} \in \kappa \cap M$, the set

$$D_{\bar{\alpha}} = \big\{ r \in \mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}} \mid r \Vdash \text{“}\dot{x} \notin [\dot{T}_\alpha]\text{”} \vee \exists u \, \big[\mathrm{lh}(u) = \bar{\alpha} \ \wedge \ r \Vdash \text{“}\check{u} \subseteq \dot{x} \ \wedge \ \check{u} \in \dot{T}_\alpha\text{”} \big] \big\}$$

is an open dense subset of $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$ and an element of M. Hence we can find $r_{\bar{\alpha}} \in D_{\bar{\alpha}} \cap M$ with $\langle p_*, \dot{q}_* \rangle \leq_{\mathbb{C}_{\vec{s}}(A)} r_{\bar{\alpha}}$ and $u_{\bar{\alpha}}$ with $\mathrm{lh}(u_{\bar{\alpha}}) = \bar{\alpha}$ and $r_{\bar{\alpha}} \Vdash$ "$\check{u}_{\bar{\alpha}} \subseteq \dot{x} \wedge \check{u}_\alpha \in \dot{T}_\alpha$". Define $u = \bigcup\{u_{\bar{\alpha}} \mid \bar{\alpha} < \alpha_{p_\eta}\}$. Then $\langle p_*, \dot{q}_* \rangle \Vdash$ "$\check{u} \subseteq \dot{x}$" and hence $u \in T_\alpha$.

Next, if $y \in A \cap M$, then the set

$$E_y = \{ \langle p, \dot{q} \rangle \in \mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}} \mid y \in a_p \ \wedge \big(r \Vdash \text{“}\dot{x} = \check{y}\text{”} \ \vee \ \exists \delta < \kappa \ r \Vdash \text{“}\dot{x} \restriction \check{\delta} \neq \check{y} \restriction \check{\delta}\text{”} \big) \}$$

is an element of M and a open dense subset of $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$. This shows that for every $y \in A \cap M$ there is a condition $r_y \in E_y \cap M$ with $\langle p_*, \dot{q}_* \rangle \leq_{\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}} r_y$. Having established that $a_{p_*} \subseteq M$ above, we can conclude that $a_{p_*} = A \cap M$ and $u \nsubseteq y$ for every $y \in A \cap M$.

By our assumptions on $\vec{s}$, there is an $\alpha_{p_*} \leq \alpha_* < \kappa$ with $u = s_{\alpha_*}$ and we can find a condition $\bar{p}$ in $\mathbb{C}_{\vec{s}}(A)$ with $\bar{p} \leq_{\mathbb{C}_{\vec{s}}(A)} p_*$, $\alpha_* < \alpha_{\bar{p}}$, $a_{\bar{p}} = a_{p_*}$ and $t_{\bar{p}}(\alpha_*) = 0$. Let $\bar{G} * \bar{H}$ be $(\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic over V with $\langle \bar{p}, \dot{q}_* \rangle \in \bar{G} * \bar{H}$. Then $\dot{x}^{\bar{G}} \in [\dot{T}^{\bar{G}}_\alpha]^{\mathrm{V}[\bar{G}, \bar{H}]}$ and hence $u = \dot{x}^{\bar{G}} \restriction \mathrm{lh}(u) \in \dot{T}^{\bar{G}}_\alpha$. But we also have $\alpha_* \geq \alpha_{p_*} > \alpha$, $s_{\alpha_*} = u$ and $t_{\bar{G}}(\alpha_*) = t_{\bar{p}}(\alpha_*) = 0$, contradicting the definition of $\dot{T}_\alpha$. □

Corollary 4.2. *Assume that $\eta^{<\eta} < \kappa$ holds for every $\eta < \kappa$. If S is a fat stationary subset of κ, $\dot{\mathbb{Q}}$ is a $\mathbb{C}_{\vec{s}}(A)$-name for a strongly S-complete partial order and $G * H$ is $(\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic over V, then the sequence $\langle \dot{T}^G_\alpha \mid \alpha < \kappa \rangle$ witnesses that A is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G, H]$.*

Proof. Pick θ satisfying Lemma 3.5.(f) and some $x \in \mathrm{H}(\theta)$. By Lemma 3.5, there is a dense set of conditions $\langle p, \dot{q} \rangle$ in $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$ such that the statements (i)-(iv) listed in the lemma hold for some elementary submodel M of $\mathrm{H}(\theta)$ of cardinality less than κ with $x \in M$. Since every such condition p is the infimum of a sequence of conditions in $\mathbb{C}_{\vec{s}}(A) \cap M$, Proposition 2.2 shows that $a_p \subseteq M$ holds for these conditions. We can conclude that the assumptions of Lemma 4.1 are satisfied in this case. □

We close this section with another corollary of Lemma 4.1 that directly implies the statement of Theorem 2.3. Note that, in contrast to the last corollary, the following argument only uses the assumption that κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

Corollary 4.3. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{C}_{\vec{s}}(A)$-name for a σ-closed, $<\kappa$-distributive partial order. If $G * H$ is $(\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic over V, then the sequence $\langle \dot{T}^G_\alpha \mid \alpha < \kappa\rangle$ witnesses that A is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G, H]$.*

Proof. We show that the assumptions of Lemma 4.1 are satisfied in this setting. Let θ be a sufficiently large regular cardinal, $x \in \mathrm{H}(\theta)$ and r be a condition in $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$. Since our assumptions imply that the partial order $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$ is $<\kappa$-distributive and therefore the intersection of less than κ-many open dense subsets of this partial order is nonempty, we can simultaneously construct a $\subseteq$-increasing sequence $\langle M_n \mid n < \omega\rangle$ of elementary submodels of $\mathrm{H}(\theta)$ of cardinality less than κ and a descending sequence $\langle\langle p_n, \dot{q}_n\rangle \mid n < \omega\rangle$ of conditions in $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$ below r such that the following statements hold for every $n < \omega$.

(i) $r, x, \mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}} \in M_0$, $\langle p_n, \dot{q}_n\rangle, M_n \in M_{n+1}$ and $\kappa \cap M_n \in \kappa$.
(ii) $\langle p_n, \dot{q}_n\rangle$ is an element of every open dense subset of $\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}}$ that is an element of M_n.

Set $M = \bigcup_{n<\omega} M_n$ and let p be the infimum of the sequence $\langle p_n \mid n < \omega\rangle$ in $\mathbb{C}_{\vec{s}}(A)$. By our assumption, there is a $\mathbb{C}_{\vec{s}}(A)$-name $\dot{q}$ for a condition in $\dot{\mathbb{Q}}$ with $\langle p, \dot{q}\rangle \leq_{\mathbb{C}_{\vec{s}}(A)*\dot{\mathbb{Q}}} \langle p_n, \dot{q}_n\rangle$ for all $n < \omega$. Then $\langle p, \dot{q}\rangle$ is strongly $(M, \mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic with the desired properties. □

5. The Main Forcing Construction

In this section and the next, we work under the following assumptions.

(1) κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $\eta^{<\eta} < \kappa$ for every $\eta < \kappa$.
(2) $\lambda = 2^\kappa$ is a regular cardinal.
(3) $\langle S_\alpha \mid \alpha \leq \kappa\rangle$ is a sequence of disjoint stationary subsets of S^κ_η if κ is the successor of a regular cardinal η and a sequence of disjoint fat stationary subsets of κ if κ is inaccessible.

We define $S = S_\kappa$, $\vec{S} = \langle S_\alpha \mid \alpha < \kappa\rangle$ and $\tilde{S}_\alpha = \kappa \setminus S_\alpha$ for every $\alpha < \kappa$.

Note that assumption (1) implies that κ is either an inaccessible cardinal or the successor of a regular cardinal. In the following, we will work with stationary subsets R of κ such that either $\kappa = \eta^+$ and $R \subseteq S^\kappa_\eta$ or κ is inaccessible and R is fat stationary. We write $R^* = R \cup S^\kappa_{<\eta}$ in the first case and $R^* = R$ in the second case. By using [1, Lemma 1.2], it is easy to see that R^* is a fat stationary subset of κ in both cases.

Towards a proof of Theorem 1.4, we recursively define a forcing that simultaneously performs the following three tasks.[7]

- Generically add a sequence $\vec{A} = \langle A_\delta \mid \delta < 2^\kappa\rangle$ of subsets of κ in the generic extension $\mathrm{V}[G]$ such that every element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ is coded (in a sense made precise later on) by exactly one A_δ.
- Generically code $\vec{A}$ to ensure that this sequence is definable over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ by a Σ_1-formula using a parameter $y \subseteq \kappa$ that is added by our forcing.
- Ensure that the parameter y is definable in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ by a Σ_1-formula that uses the sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ as a parameter.

In this situation, we can well-order $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ in the desired way by identifying each element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ with the unique A_δ coding it. The generic coding used in this construction will be a variation of the *almost disjoint coding forcing* from Section 2 that was introduced in [2, Section 2] and combines the original forcing with iterated club shooting. The additional coding to make the parameter y definable from $\langle S_\alpha \mid \alpha < \kappa\rangle$ will be achieved by further iterated club shooting.

Before we begin with the construction of our forcing, we specify a number of notions used in this construction and fix some more assumptions. We start with several notions of coding sets into other sets.

- We let $\prec\cdot,\cdot\succ : \mathrm{On} \times \mathrm{On} \longrightarrow \mathrm{On}$ denote the *Gödel pairing function*.
- We say that $A \subseteq \kappa$ *codes an element* z *of* $\mathrm{H}(\kappa^+)$ if there is a bijection $b : \kappa \longrightarrow \mathrm{tc}(\{z\})$ such that

$$A = \{\prec 0, \prec\alpha,\beta\succ\succ \mid \alpha,\beta < \kappa,\ b(\alpha) \in b(\beta)\} \cup \{\prec 1,\alpha\succ \mid \alpha < \kappa,\ b(\alpha) \in z\}.$$

Note that z and b are uniquely determined by A.

[7]This extends the construction of the forcing to witness Theorem 1.3, as provided in [11, Section 2], by the additional third task below. Note that this task introduces the additional technical difficulty that the witnessing forcing for Theorem 1.4 cannot be (unlike the witnessing forcing for Theorem 1.3) $<\kappa$-closed. The final forcing to witness Theorem 1.4 will be a two-step iteration of the forcing described in this section preceded by a preliminary forcing to achieve assumption (5) below. This two-step iteration is described in detail at the end of Section 6.

- Given $x, y \in {}^{\kappa}\kappa$, we define $x \oplus y \in {}^{\kappa}\kappa$ by setting

$$(x \oplus y)\ (\alpha) := \begin{cases} x(\beta), & \text{if } \alpha = \prec 0, \beta \succ, \\ y(\beta), & \text{if } \alpha = \prec 1, \beta \succ, \\ 0, & \text{otherwise.} \end{cases}$$

  for all $\alpha < \kappa$.
- Given $\alpha, \beta < \kappa$, we define $c(\alpha, \beta) \in {}^{\kappa}2$ by setting

$$c(\alpha, \beta)\ (\gamma) := \begin{cases} 1, & \text{if } \gamma \in \{\prec 0, \alpha \succ, \prec 1, \beta \succ\}, \\ 0, & \text{otherwise.} \end{cases}$$

  for all $\gamma < \kappa$.

In addition to our previous assumptions, we also assume that the following objects exist. Note that we can (and will) achieve (5) by a preparatory almost disjoint coding forcing, using Corollary 4.2.

(4) $\vec{w} = \langle w_\gamma \mid \gamma < \lambda \rangle$ is a sequence of pairwise distinct elements of ${}^{\kappa}2$.

(5) $\vec{T} = \langle T_\alpha \mid \alpha < \kappa \rangle$ is a sequence of subtrees of ${}^{<\kappa}\kappa$ with the property that

$$\{w_{\bar{\gamma}} \oplus w_\gamma \mid \bar{\gamma} < \gamma < \lambda\} \ = \ \bigcup_{\alpha < \kappa} [T_\alpha]^{\mathrm{V}[G]} \tag{1}$$

holds whenever $\mathrm{V}[G]$ is a generic extension of the ground model V by a strongly S^*-complete forcing.

$\vec{T}$ thus witnesses that $\{w_{\bar{\gamma}} \oplus w_\gamma \mid \bar{\gamma} < \gamma < \lambda\}$ is a $\mathbf{\Sigma}^0_2$-subset of ${}^{\kappa}\kappa$ in every generic extension by a strongly S^*-complete forcing.

In the following, we inductively construct a sequence $\vec{\mathbb{P}}_{\vec{w}} = \langle \mathbb{P}_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders such that $\mathbb{P}_\delta$ is a complete subforcing of $\mathbb{P}_\gamma$ whenever $\delta < \gamma \leq \lambda$. Fix $\gamma \leq \lambda$ and assume that we constructed $\mathbb{P}_\delta$ with that property for every $\delta < \gamma$.

Definition 5.1. We call a tuple

$$p \ = \ \langle s_p, t_p, \vec{d}_p, \vec{c}_p, \vec{A}_p \rangle$$

a $\mathbb{P}_\gamma$-*candidate* if the following statements hold for some ordinals $\beta_p < \kappa$ and $\gamma_p < \min\{\gamma + 1, \lambda\}$.

(i) $s_p : \beta_p + 1 \longrightarrow {}^{<\kappa}2$.

(ii) $t_p : \beta_p + 1 \longrightarrow 2$.

(iii) $\vec{d}_p = \langle d_{p,\alpha} \mid \alpha \leq \beta_p \rangle$ is such that $d_{p,\alpha}$ is a closed subset of $\tilde{S}_\alpha \cap (\beta_p + 1)$ for every $\alpha \leq \beta_p$. We require that $d_{p,\alpha}$ is the empty set if $\alpha = \beta \cdot 7 + i$ with $i < 7$, $\beta = \prec \gamma, \delta \succ$ and one of the following statements holds.[8]

[8]We will shoot clubs through the $\tilde{S}_\alpha$, but only for specific α, in order to ensure that in the end, certain sets will be definable using $\vec{S}$ as parameter by checking which of the S_α remained stationary. Coding each piece of information both *positively* and *negatively*

(a) $i < 2$ and $s_p(\gamma)(\delta) = i$.
(b) $i = 2$ and $\mathrm{lh}(s_p(\gamma)) \leq \delta$.
(c) $2 < i < 5$ and $t_p(\beta) = i - 3$.
(d) $i = 5$ and $s_p(\gamma) \notin T_\delta$.
(e) $i = 6$ and $s_p(\gamma) \in T_\delta$.

(iv) $\vec{c}_p = \langle c_{p,x} \mid x \in a_p \rangle$ is a sequence that satisfies the following properties.
(a) a_p is a subset of $\{w_\delta \oplus c(\alpha, i) \mid \delta < \gamma_p,\ \alpha < \kappa,\ i < 2\}$ of cardinality less than κ.
(b) If $x \in a_p$, then $c_{p,x}$ is a closed subset of $\beta_p + 1$ and the implication

$$s_p(\alpha) \subseteq x \ \longrightarrow\ t_p(\alpha) = 1$$

holds for every $\alpha \in c_{p,x}$.

(v) $\vec{A}_p = \langle \dot{A}_{p,\delta} \mid \delta < \gamma_p \rangle$ is a sequence that satisfies the following statements.
(a) If $\delta < \gamma_p$, then $\dot{A}_{p,\delta}$ is a $\mathbb{P}_\delta$-nice name for a subset of κ (and thus by our assumptions a $\mathbb{P}_{\tilde{\delta}}$-nice name for a subset of κ for every $\delta \leq \tilde{\delta} < \gamma$).
(b) If $\bar{\gamma} < \gamma_p$ and G is $\mathbb{P}_{\bar{\gamma}}$-generic over the ground model V, then either $|\lambda|^{\mathrm{V}[G]} = |\bar{\gamma}|^{\mathrm{V}[G]}$ holds[9] or in V[G], there is a sequence $\langle y_\delta \mid \delta \leq \bar{\gamma} \rangle$ of pairwise distinct elements of $\mathrm{H}(\kappa^+)$ s.t. $\dot{A}^G_{p,\delta}$ codes y_δ for every $\delta \leq \bar{\gamma}$.

Given a $\mathbb{P}_\gamma$-candidate p and $\delta \leq \gamma$, we define $p \restriction \delta$ to be the tuple

$$\langle s_p, t_p, \vec{d}_p, \langle c_{p,x} \mid x \in a_p \restriction \delta \rangle, \vec{A}_p \restriction \min\{\gamma_p, \delta\} \rangle,$$

where $a_p \restriction \delta = a_p \cap \{w_{\bar{\delta}} \oplus c(\alpha, i) \mid \bar{\delta} < \delta,\ \alpha < \kappa,\ i < 2\}$.

It is straightforward to check that whenever p is a $\mathbb{P}_\gamma$-candidate and $\delta \leq \gamma$, $p \restriction \delta$ is a $\mathbb{P}_\delta$-candidate.

Definition 5.2. A $\mathbb{P}_\gamma$-candidate p is a condition in $\mathbb{P}_\gamma$ if the following statement holds for all $\delta < \gamma_p$, $\alpha < \kappa$ and $i < 2$ with $w_\delta \oplus c(\alpha, i) \in a_p$.

(vi) If $p \restriction \delta$ is a condition in $\mathbb{P}_\delta$, then

$$p \restriction \delta \ \Vdash_{\mathbb{P}_\delta} \ \text{“}i = 1 \ \longleftrightarrow \ \check{\alpha} \in \dot{A}_{p,\delta}\text{”}.^{10}$$

will make sure that the complexity of those definitions will in fact be Δ_1-definable. Note that each $\tilde{S}_\alpha$ is a fat stationary subset of κ.

[9]We will show later that this case never occurs (see Corollary 5.12).

[10]The idea behind this construction is that the set a_p collects information about the interpretations of names in $\vec{A}_p$ that is already decided by the condition p. This will allow us to use the almost disjoint coding part of the forcing (see Clause (iv), (b)) to add a subset of κ that in the end codes $\bigcup_{p \in G} a_p$ and thus also $\bigcup_{p \in G} \vec{A}_p$ whenever G is

Given conditions p and q in $\mathbb{P}_\gamma$, we define $q \leq_{\mathbb{P}_\gamma} p$ to hold if $s_p = s_q \restriction (\beta_p + 1)$, $t_p = t_q \restriction (\beta_p + 1)$, $d_{p,\alpha} = d_{q,\alpha} \restriction (\beta_p + 1)$ for every $\alpha \leq \beta_p$, $a_p \subseteq a_q$, $\vec{A}_p = \vec{A}_q \restriction \gamma_p$ and $c_{p,x} = c_{q,x} \restriction (\beta_q + 1)$ for every $x \in a_p$.

Proposition 5.3. *If p is a condition in $\mathbb{P}_\gamma$ and $\delta < \gamma$, then $p \restriction \delta$ is a condition in $\mathbb{P}_\delta$. In particular, every condition p in $\mathbb{P}_\gamma$ is also a condition in $\mathbb{P}_{\gamma_p}$.*

Proof. Let $\delta < \gamma$ and assume that $p \restriction \bar{\delta}$ is a condition in $\mathbb{P}_{\bar{\delta}}$ for every $\bar{\delta} < \delta$. Fix $\bar{\delta} < \delta$, $\alpha < \kappa$ and $i < 2$ with $w_{\bar{\delta}} \oplus c(\alpha, i) \in a_{p \restriction \delta}$. Then $(p \restriction \delta) \restriction \bar{\delta} = p \restriction \bar{\delta}$ is a condition in $\mathbb{P}_{\bar{\delta}}$ and $a_{p \restriction \delta} = a_p \restriction \delta \subseteq a_p$. Since p is a condition in $\mathbb{P}_\gamma$, this implies $\bar{\delta} < \gamma_p$ and

$$(p \restriction \delta) \restriction \bar{\delta} \Vdash_{\mathbb{P}_\delta} \text{“} i = 1 \longleftrightarrow \check{\alpha} \in \dot{A}_{p,\bar{\delta}} \text{”}.$$

We can conclude that $p \restriction \delta$ is a condition in $\mathbb{P}_\delta$. □

The following statement is a direct consequence of the above definitions.

Proposition 5.4. *If p is a condition in $\mathbb{P}_\gamma$ and $\vec{A}$ is a sequence of length smaller than $\min\{\gamma+1, \lambda\}$ such that $\vec{A}_p \subseteq \vec{A}$ and $\vec{A}$ satisfies the statements listed in Clause (v) of Definition 5.1, then the tuple $\langle s_p, t_p, \vec{d}_p, \vec{c}_p, \vec{A} \rangle$ is a condition in $\mathbb{P}_\gamma$ that is stronger than p.* □

Proposition 5.5. *If $\bar{\gamma} < \min\{\gamma + 1, \lambda\}$, then the set of all conditions p in $\mathbb{P}_\gamma$ with $\gamma_p \geq \bar{\gamma}$ is dense in $\mathbb{P}_\gamma$.*

Proof. Fix a condition p in $\mathbb{P}_\gamma$ with $\gamma_p < \bar{\gamma}$. Since $\bar{\gamma} < \lambda = 2^\kappa$, we can recursively construct a sequence $\vec{A}$ of length $\bar{\gamma}$ that satisfies the statements listed in Clause (v) of Definition 5.1. By Proposition 5.4, the resulting tuple $\langle s_p, t_p, \vec{d}_p, \vec{c}_p, \vec{A} \rangle$ is a condition in $\mathbb{P}_\gamma$ that is stronger than p. □

Lemma 5.6. *If $\delta < \gamma$, then $\mathbb{P}_\delta$ is a complete subforcing of $\mathbb{P}_\gamma$.*

Proof. Every condition in $\mathbb{P}_\delta$ is a condition in $\mathbb{P}_\gamma$, $\leq_{\mathbb{P}_\delta} = \leq_{\mathbb{P}_\gamma} \restriction (\mathbb{P}_\delta \times \mathbb{P}_\delta)$ and, if q is a condition in $\mathbb{P}_\delta$ and p is a condition in $\mathbb{P}_\gamma$ with $p \leq_{\mathbb{P}_\gamma} q$, then Proposition 5.3 shows that $p \restriction \delta$ is a condition in $\mathbb{P}_\delta$ and it is easy to check that $p \restriction \delta \leq_{\mathbb{P}_\delta} q$ holds. Hence it suffices to show that every maximal antichain in $\mathbb{P}_\delta$ is maximal in $\mathbb{P}_\gamma$.

Fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}_\delta$ and a condition p_0 in $\mathbb{P}_\gamma$. By Proposition 5.5, there is a condition p with $p \leq_{\mathbb{P}_\gamma} p_0$ and $\gamma_p \geq \delta$. Proposition 5.3 implies

$\mathbb{P}_\lambda$-generic. In Clause (iii), we simultaneously work on making this subset of κ lightface definable.

that $p \restriction \delta$ is a condition in $\mathbb{P}_\delta$. Hence we find a condition q in $\mathbb{P}_\delta$ and $r \in \mathcal{A}$ with $q \leq_{\mathbb{P}_\delta} p \restriction \delta, r$. Then $\gamma_q = \delta$. Define p^* to be the tuple

$$\langle s_q, t_q, \vec{d}_q, \langle c_{p,x} \mid x \in a_p \setminus a_q \rangle \ \cup \ \langle c_{q,x} \mid x \in a_q \rangle, \vec{A}_p \rangle.$$

Then p^* is a $\mathbb{P}_\gamma$-candidate with $\gamma_{p^*} = \gamma_p$. Fix $\bar{\delta} < \gamma$, $\alpha < \kappa$ and $i < 2$ such that $p^* \restriction \bar{\delta}$ is a condition in $\mathbb{P}_{\bar{\delta}}$ and $x = w_{\bar{\delta}} \oplus c(\alpha, i) \in a_p \cup a_q$. If $x \in a_q$, then $\bar{\delta} < \delta \leq \gamma_{p^*}$ and $\vec{A}_q = \vec{A}_p \restriction \delta$ implies that $p^* \restriction \bar{\delta} \leq_{\mathbb{P}_{\bar{\delta}}} q \restriction \bar{\delta}$. Hence

$$p^* \restriction \bar{\delta} \Vdash_{\mathbb{P}_{\bar{\delta}}} \text{“} i = 1 \longleftrightarrow \check{\alpha} \in \dot{A}_{p,\bar{\delta}} \text{”}$$

holds in this case. Now assume that $x \in a_p \setminus a_q$. Since $q \leq_{\mathbb{P}_\delta} p \restriction \delta$, we have $p^* \restriction \bar{\delta} \leq_{\mathbb{P}_{\bar{\delta}}} p \restriction \bar{\delta}$ and this implies that the above forcing statement also holds in this case. Therefore p^* is a condition in $\mathbb{P}_\gamma$ and our construction ensures that $p^* \leq_{\mathbb{P}_\gamma} p, q$ holds. Hence $\mathcal{A}$ is a maximal antichain in $\mathbb{P}_\gamma$. □

This completes the construction of the sequence $\vec{\mathbb{P}}_{\vec{w}}$ of partial orders.

Proposition 5.7. *Let $\gamma \leq \lambda$, $\bar{\lambda} < \lambda$ and $\langle p_\alpha \mid \alpha < \bar{\lambda} \rangle$ be a sequence of conditions in $\mathbb{P}_\gamma$ such that $\vec{A}_{p_\alpha} \subseteq \vec{A}_{p_\beta}$ holds for all $\alpha < \beta < \bar{\lambda}$. Then $\vec{A} = \bigcup \{ \vec{A}_{p_\alpha} \mid \alpha < \bar{\lambda} \}$ satisfies the statements listed in Clause (v) of Definition 5.1.* □

Proposition 5.8. *If $\bar{\beta} < \kappa$ and $\gamma \leq \lambda$, then the set of all conditions q in $\mathbb{P}_\gamma$ with $\beta_q \geq \bar{\beta}$ is dense in $\mathbb{P}_\gamma$. In particular, if $\gamma \leq \lambda$ and G is $\mathbb{P}_\gamma$-generic over V, then $\kappa = \sup\{\beta_p \mid p \in G\}$.*

Proof. Fix a condition p in $\mathbb{P}_\gamma$ with $\beta_p < \bar{\beta}$ and define q to be the tuple

$$\langle s_p \cup \langle (\alpha, \emptyset) \mid \beta_p \leq \alpha \leq \bar{\beta} \rangle, t_p \cup \langle (\alpha, 1) \mid \beta_p \leq \alpha \leq \bar{\beta} \rangle, \langle d_{q,\alpha} \mid \alpha \leq \bar{\beta} \rangle, \vec{c}_p, \vec{A}_p \rangle$$

with $d_{q,\alpha} = d_{p,\alpha}$ for $\alpha \leq \beta_p$ and $d_{q,\alpha} = \emptyset$ for all $\beta_p < \alpha \leq \bar{\beta}$. Then it is easy to see that q is a condition in $\mathbb{P}_\gamma$ with $q \leq_{\mathbb{P}_\gamma} p$ and $\beta_q = \bar{\beta}$. □

Lemma 5.9. *If $\gamma \leq \lambda$, then $\mathbb{P}_\gamma$ is strongly S^*-complete. Moreover, if p is a condition in $\mathbb{P}_\gamma$ and $\zeta \leq \beta_p$ is such that $d_{p,\zeta}$ is required to be the empty set by one of the statements listed in Clause (iii) of Definition 5.1, then the partial order of conditions in $\mathbb{P}_\gamma$ below p is strongly S^*_ζ-complete.*

Proof. Let $\vec{D} = \langle D_\alpha \mid \alpha < \kappa \rangle$ be defined by setting $D_\alpha = \{q \in \mathbb{P}_\gamma \mid \beta_q \geq \alpha\}$ for every $\alpha < \kappa$. Each D_α is open dense in $\mathbb{P}_\gamma$ by Proposition 5.8. We show that $\vec{D}$ witnesses that $\mathbb{P}_\gamma$ is strongly S^*-complete.

Assume $\theta > \kappa$ is a regular cardinal with $\mathcal{P}(\mathbb{P}_\gamma) \in \mathrm{H}(\theta)$, M is an elementary substructure of $\mathrm{H}(\theta)$ of cardinality less than κ with $\beta = \sup(M \cap \kappa) \in S^*$ and $\vec{D}, \mathbb{P}_\gamma \in M$ and $\vec{p} = \langle p_\xi \mid \xi < \eta \rangle \subseteq M$ is a descending sequence of conditions in $\mathbb{P}_\gamma$ such that $\{\alpha < \beta \mid \exists \xi < \eta \; p_\xi \in D_\alpha\}$ is unbounded in β. This implies that $\beta = \sup_{\xi<\eta} \beta_{p_\xi} \in S^*$. We define a tuple

$$p_* \; = \; \langle s, t, \langle d_\alpha \mid \alpha \leq \beta \rangle, \langle c_x \mid x \in a \rangle, \vec{A} \rangle,$$

by setting

- $s \; = \; \{\langle \beta, \emptyset \rangle\} \; \cup \; \bigcup \{s_{p_\xi} \mid \xi < \eta\}$.
- $t \; = \; \{\langle \beta, 1 \rangle\} \; \cup \; \bigcup \{t_{p_\xi} \mid \xi < \eta\}$.
- $\bar{d}_\alpha \; = \; \bigcup \{d_{p_\xi, \alpha} \mid \xi < \eta, \; \alpha \leq \beta_{p_\xi}\}$ for every $\alpha < \beta$.
- $d_\alpha \; = \; \begin{cases} \bar{d}_\alpha \cup \{\beta\}, & \text{if } \alpha < \beta \text{ and } \bar{d}_\alpha \neq \emptyset. \\ \emptyset, & \text{if either } \alpha = \beta \text{ or } \alpha < \beta \text{ and } \bar{d}_\alpha = \emptyset. \end{cases}$
- $a \; = \; \bigcup \{a_{p_\xi} \mid \xi < \eta\}$,
- $c_x \; = \; \{\beta\} \; \cup \; \bigcup \{c_{p_\xi, x} \mid \xi < \eta, \; x \in a_{p_\xi}\}$ for each $x \in a$,
- $\vec{A} \; = \; \bigcup \{\vec{A}_{p_\xi} \mid \xi < \eta\}$.

By Proposition 5.7, $\vec{A}$ satisfies the statements listed in Clause (v) of Definition 5.1. Since $\beta \in S^* \subseteq \tilde{S}_\alpha$ for every $\alpha < \beta$, we can conclude that p_* is a $\mathbb{P}_\gamma$-candidate. Fix $\delta < \gamma$, $\nu < \kappa$ and $i < 2$ with $x = w_\delta \oplus c(\nu, i) \in a$. Then there is $\xi < \eta$ with $x \in a_{p_\xi}$ and hence $\delta < \gamma_{p_\xi} \leq \gamma_{p_*}$. If $p_* \restriction \delta$ is a condition in $\mathbb{P}_\delta$, then $p_* \restriction \delta \leq_{\mathbb{P}_\delta} p_\xi \restriction \delta$ and hence it forces that α is an element of $\dot{A}_{p_*, \delta}$ if and only if $i = 1$. This shows that p_* is a condition in $\mathbb{P}_\gamma$ and our construction ensures that $p_* \leq_{\mathbb{P}_\gamma} p_\xi$ holds for every $\xi < \eta$.

The proof of the second statement is similar, noting that if p and ζ are as in its hypothesis, then we have $S^*_\zeta \subseteq \tilde{S}_\alpha$ for all $\zeta \neq \alpha < \kappa$ and $d_{q,\zeta} = \emptyset$ for every condition q in $\mathbb{P}_\gamma$ below p. Thus, if θ, M, β and $\vec{p}$ satisfy the above statements with S^* replaced by S^*_ζ, then we can repeat the above construction to obtain a condition p_* witnessing strong S^*_ζ-completeness. □

If $\kappa = \eta^+$ is a successor cardinal, then the assumption $S_\alpha \subseteq S^\kappa_\eta$ can be used to show that $\mathbb{P}_\gamma$ is in fact $<\eta$-closed.

Lemma 5.10. *If $\gamma < \lambda$ and $p \in \mathbb{P}_\gamma$ is a condition with $\gamma_p = \gamma$, then the forcing $\mathbb{P}_\gamma$ satisfies the κ^+-cc below p.*

Proof. By a standard Δ-system argument, using the assumption that $\kappa = \kappa^{<\kappa}$ and that whenever $q \leq p$ in $\mathbb{P}_\gamma$ we have that $\vec{A}_q = \vec{A}_p$. □

Lemma 5.11. *If q is a condition in $\mathbb{P}_\lambda$ and $\mathcal{D}$ is a collection of less than λ-many open dense subsets of $\mathbb{P}_\lambda$, then there is a condition p in $\mathbb{P}_\lambda$ such that $p \leq_{\mathbb{P}_\lambda} q$ and the set $D \cap \mathbb{P}_{\gamma_p}$ is dense below p in $\mathbb{P}_{\gamma_p}$ for every $D \in \mathcal{D}$.*

Proof. We start by proving the following claim. An iterated application of this claim will yield the statement of the lemma.

Claim. *Let q_0 be a condition in $\mathbb{P}_\lambda$ and D be a open dense subset of $\mathbb{P}_\lambda$. Then there is a condition q_0^* in $\mathbb{P}_\lambda$ such that $q_0^* = \langle s_{q_0}, t_{q_0}, \vec{d}_{q_0}, \vec{c}_{q_0}, \vec{A}_{q_0^*}\rangle \leq_{\mathbb{P}_\lambda} q_0$ and $D \cap \mathbb{P}_{\gamma_{q_0^*}}$ is dense below q_0^* in $\mathbb{P}_{\gamma_{q_0^*}}$.*

Proof of the Claim. We inductively construct a sequence $\langle q_\alpha \mid 0 < \alpha < \theta\rangle$ of incompatible conditions below q_0 in $\mathbb{P}_\lambda$ with $0 < \theta \leq \kappa^+$ and $\vec{A}_{q_{\bar{\alpha}}} \subseteq \vec{A}_{q_\alpha}$ for all $\bar{\alpha} < \alpha < \theta$: Assume that the sequence $\langle q_{\bar{\alpha}} \mid 0 < \bar{\alpha} < \alpha\rangle$ is already constructed. If there is $p_\alpha \in D$ such that $p_\alpha \leq_{\mathbb{P}_\lambda} \langle s_{q_0}, t_{q_0}, \vec{d}_{q_0}, \vec{c}_{q_0}, \bigcup_{\bar{\alpha}<\alpha} \vec{A}_{p_{\bar{\alpha}}}\rangle$ and the conditions p_α and $q_{\bar{\alpha}}$ are incompatible in $\mathbb{P}_\lambda$ for all $0 < \bar{\alpha} < \alpha$, then we set $q_\alpha = p_\alpha$ and we continue our construction. Otherwise, we stop our construction and set $\theta = \alpha$.

Define $\vec{A} = \bigcup_{\alpha<\theta} \vec{A}_{q_\alpha}$ and $q_\alpha^* = \langle s_{q_\alpha}, t_{q_\alpha}, \vec{d}_{q_\alpha}, \vec{c}_{q_\alpha}, \vec{A}\rangle$ for all $\alpha < \theta$. Given $\alpha < \theta$, Proposition 5.7 shows that q_α^* is a condition in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^* and q_α. In particular, the set $\mathcal{A} = \{q_\alpha^* \mid 0 < \alpha < \theta\}$ is an antichain in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^*. By Lemma 5.10, this means that the above construction has stopped at stage $\theta < \kappa^+$, because no suitable condition p_θ could be found. This implies that $\mathcal{A}$ is a *maximal* antichain in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^*.

Pick a condition p in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^*. Then there is $0 < \alpha < \theta$ and a condition r in $\mathbb{P}_{\gamma_{q_0^*}}$ with $r \leq_{\mathbb{P}_{\gamma_{q_0^*}}} p, q_\alpha^*$. Since q_α^* is an element of D, we get $r \in D$. This shows that the condition q_0^* has the desired properties. □

Let $\langle D_\alpha \mid \alpha < \bar{\lambda}\rangle$ be an enumeration of $\mathcal{D}$ such that $\bar{\lambda} < \lambda$ is a limit ordinal. By the above claim and Proposition 5.7, we can construct a decreasing sequence $\langle q_\alpha \mid \alpha \leq \bar{\lambda}\rangle$ of conditions in $\mathbb{P}_\lambda$ such that $q = q_0$, $q_\alpha = \langle s_q, t_q, \vec{d}_q, \vec{c}_q, \vec{A}_{q_\alpha}\rangle$ for all $\alpha \leq \bar{\lambda}$ and $D_\alpha \cap \mathbb{P}_{\gamma_{q_{\alpha+1}}}$ is dense below $q_{\alpha+1}$ in $\mathbb{P}_{\gamma_{q_{\alpha+1}}}$ for all $\alpha < \bar{\lambda}$.

Pick a condition r in $\mathbb{P}_{\gamma_{q_{\bar{\lambda}}}}$ below $q_{\bar{\lambda}}$ and $\alpha < \bar{\lambda}$. Then our construction ensures $\vec{A}_r = \vec{A}_{q_{\bar{\lambda}}}$ and $r \restriction \gamma_{q_{\alpha+1}} \leq_{\mathbb{P}_{\gamma_{q_{\alpha+1}}}} q_{\bar{\lambda}} \restriction \gamma_{q_{\alpha+1}} = q_{\alpha+1}$. This allows us to find a condition $\bar{r}_\alpha \in D_\alpha$ with $\bar{r}_\alpha \leq_{\mathbb{P}_{\gamma_{q_{\alpha+1}}}} r \restriction \gamma_{q_{\alpha+1}}$. We define $\vec{c} = \langle c_x \mid x \in a_r \cup a_{\bar{r}_\alpha}\rangle$ by letting $c_x = c_{\bar{r}_\alpha,x}$ if $x \in a_{\bar{r}_\alpha}$ and letting $c_x = c_{r,x}$ otherwise. Then $r_\alpha = \langle s_{\bar{r}_\alpha}, t_{\bar{r}_\alpha}, \vec{d}_{\bar{r}_\alpha}, \vec{c}, \vec{A}_r\rangle$ is a $\mathbb{P}_{\gamma_{q_{\bar{\lambda}}}}$-candidate with $\bar{r}_\alpha = r_\alpha \restriction \gamma_{q_{\alpha+1}}$. Moreover, if $\delta < \gamma_{q_{\bar{\lambda}}}$ and $r_\alpha \restriction \delta$ is a condition in $\mathbb{P}_\delta$, then this condition is stronger than $r \restriction \delta$. We can conclude that r_α is actually a condition in $\mathbb{P}_{\gamma_{q_{\bar{\lambda}}}}$ that is a common extension of r and $\bar{r}_\alpha$, and an element of D_α. This shows that $p = q_{\bar{\lambda}}$ has the desired properties. □

Corollary 5.12. *Forcing with $\mathbb{P}_\lambda$ preserves all cofinalities less than or equal to λ.*

Proof. By Lemma 3.2 and Lemma 5.9, forcing with $\mathbb{P}_\lambda$ preserves cofinalities less than or equal to κ. Let $\gamma \leq \lambda$ be a limit ordinal with $\mathrm{cof}(\gamma) > \kappa$ and let ν

be a regular cardinal with $\kappa \leq \nu < \mathrm{cof}(\gamma)$. Assume, towards a contradiction, that there is $q \in \mathbb{P}_\lambda$ and a $\mathbb{P}_\lambda$-name $\dot{c}$ with $q \Vdash_{\mathbb{P}_\lambda}$ "$\dot{c}: \check{\nu} \longrightarrow \check{\gamma}$ *is cofinal*". Given $\alpha < \nu$, define

$$D_\alpha = \{p \in \mathbb{P}_\lambda \mid \exists \beta < \gamma \ p \Vdash_{\mathbb{P}_\lambda} \text{"}\dot{c}(\check{\alpha}) = \check{\beta}\text{"}\}.$$

Let G be P_λ-generic over V. By Lemma 5.11, there is $p \in G$ such that $p \leq_{\mathbb{P}_\lambda} q$ and $D_\alpha \cap \mathbb{P}_{\gamma_p}$ is dense below p in $\mathbb{P}_{\gamma_p}$ for every $\alpha < \nu$. By Lemma 5.10, $\mathbb{P}_{\gamma_p}$ satisfies the κ^+-cc below p. Therefore we can define $c : \nu \longrightarrow \gamma$ in V by setting

$$c(\alpha) \ = \ \sup\{\beta + 1 \mid \exists r \in \mathbb{P}_{\gamma_p} \ [r \leq_{\mathbb{P}_{\gamma_p}} p \ \wedge \ r \Vdash_{\mathbb{P}_\lambda} \text{"}\dot{c}(\check{\alpha}) = \check{\beta}\text{"}]\}$$

for every $\alpha < \nu$. Pick $\alpha < \nu$. By Lemma 5.6, $\bar{G} = G \cap \mathbb{P}_{\gamma_p}$ is $\mathbb{P}_{\gamma_p}$-generic over V. Since $p \in \bar{G}$, the above computations show that there is an $r \in D_\alpha \cap \bar{G}$. If $\beta < \gamma$ witnesses that r is an element of D_α, then $\dot{c}^G(\alpha) = \beta < c(\alpha)$. This shows that the range of c is unbounded in γ, a contradiction. □

Corollary 5.13. *Let G be $\mathbb{P}_\lambda$-generic over* V *and A be a subset of κ in* V$[G]$. *Then there is a $\gamma < \lambda$ such that $A = \dot{A}^{G \cap \mathbb{P}_\gamma}$ for some $\mathbb{P}_\gamma$-name $\dot{A}$ for a subset of κ.*

Proof. Let $\dot{A}_0$ be a $\mathbb{P}_\lambda$-name for a subset of κ with $A = \dot{A}_0^G$ and, given $\alpha < \kappa$, let D_α be the open dense subset of $\mathbb{P}_\lambda$ consisting of all conditions in $\mathbb{P}_\lambda$ that decide the statement "$\check{\alpha} \in \dot{A}_0$". By Lemma 5.11, there is a $p \in G$ such that the set $D_\alpha \cap \mathbb{P}_{\gamma_p}$ is dense below p for every $\alpha < \kappa$. Define

$$\dot{A} \ = \ \{\langle \check{\alpha}, r \rangle \mid \alpha < \kappa, \ r \in D_\alpha \cap \mathbb{P}_{\gamma_p}, \ r \leq_{\mathbb{P}_\lambda} p, \ r \Vdash_{\mathbb{P}_\lambda} \text{"}\check{\alpha} \in \dot{A}_0\text{"}\}.$$

Then $\dot{A}$ is a $\mathbb{P}_{\gamma_p}$-name for a subset of κ and we can use Lemma 5.6 to conclude that $A = \dot{A}^G = \dot{A}^{G \cap \mathbb{P}_{\gamma_p}}$. □

6. The Proof of Theorem 1.4

We are now ready to show how the forcing constructed in the last section can be used to produce a locally Σ_1-definable well-order of $\mathrm{H}(\kappa^+)$ using only the sequence $\vec{S}$ as a parameter.

Lemma 6.1. *If G is $\mathbb{P}_\lambda$-generic over* V *and y is an element of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, *then there is a unique ordinal $\delta < \lambda$ such that for some $p \in G$ with $\delta < \gamma_p$ the set $\dot{A}^G_{p,\delta}$ codes y.*

Proof. By Corollary 5.13, there is a $\gamma < \lambda$ and a $\mathbb{P}_\gamma$-name $\dot{y}$ such that $y = \dot{y}^{G \cap \mathbb{P}_\gamma}$. Fix a condition p in $\mathbb{P}_\lambda$ with $\gamma_p \geq \gamma$. Let $\dot{A}$ be a $\mathbb{P}_{\gamma_p}$-name for a subset of κ such that the following statements hold whenever H is $\mathbb{P}_{\gamma_p}$-generic over V with $p \in H$ and $\dot{y}^H \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$.

- If there is no $\delta < \gamma_p$ such that $\dot{A}^H_{p,\delta}$ codes $\dot{y}^H$, then $\dot{A}^H$ codes $\dot{y}^H$.
- Otherwise, $\dot{A}^H$ codes an element of $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ that is not coded by some $\dot{A}^H_{p,\delta}$ with $\delta < \gamma_p$ (note that Corollary 5.12 implies that such an element always exists).

Define $\vec{A} = \vec{A}_p \cup \{\langle \gamma_p, \dot{A} \rangle\}$. Then $\vec{A}$ satisfies the statements listed in Clause (v) of Definition 5.1 and $\langle s_p, t_p, \vec{d}_p, \vec{c}_p, \vec{A} \rangle$ is a condition in $\mathbb{P}_\lambda$ below p. This density argument shows that there is $q \in G$ and $\delta < \gamma_q$ such that $\gamma_q > \gamma$ and $\dot{A}^{G \cap \mathbb{P}_{\gamma_q}}_{q,\delta} = \dot{A}^G_{q,\delta}$ codes $\dot{y}^{G \cap \mathbb{P}_{\gamma_q}} = \dot{y}^G$.

Now assume, towards a contradiction, that there are $\delta_0 < \delta_1 < \lambda$ and $p_0, p_1 \in G$ such that both $\dot{A}^G_{p_0,\delta_0}$ and $\dot{A}^G_{p_1,\delta_1}$ code y. Pick $p \in G$ with $p \leq_{\mathbb{P}_\lambda} p_0, p_1$. Then $\bar{G} = G \cap \mathbb{P}_{\delta_1}$ is $\mathbb{P}_{\delta_1}$-generic over V and Corollary 5.12 implies $|\delta_1|^{\mathrm{V}[\bar{G}]} < |\lambda|^{\mathrm{V}[\bar{G}]}$. Thus $\dot{A}^{\bar{G}}_{p,\delta_0} = \dot{A}^G_{p_0,\delta_0}$ and $\dot{A}^{\bar{G}}_{p,\delta_1} = \dot{A}^G_{p_1,\delta_1}$ code the same element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[\bar{G}]}$, contradicting Clause (v) of Definition 5.1 for the condition p. □

Corollary 6.2. *Forcing with $\mathbb{P}_\lambda$ preserves the value of 2^κ.* □

If G is $\mathbb{P}_\lambda$-generic over V, then we define

$$D(G) \ = \ \{w_\delta \oplus c(\alpha, i) \mid i < 2, \ \exists p \in G \ [\delta < \gamma_p \ \wedge \ (i = 1 \ \longleftrightarrow \ \alpha \in \dot{A}^G_{p,\delta})]\}.$$

Proposition 6.3. *If G is $\mathbb{P}_\lambda$-generic over V and $x = w_\delta \oplus c(\alpha, i) \in D(G)$, then there is a condition $p \in G$ with $x \in a_p$. In particular, $D(G) = \bigcup_{p \in G} a_p$.*

Proof. Assume $x = w_\delta \oplus c(\alpha, i) \in D(G)$. Then there is a condition $q \in G$ such that $\delta < \gamma_q$ and $q \restriction \delta \Vdash_{\mathbb{P}_\delta}$ "$i = 1 \longleftrightarrow \check{\alpha} \in \dot{A}_{q,\delta}$". We may assume that $x \notin a_q$. Fix $p_0 \in \mathbb{P}_\lambda$ with $p_0 \leq_{\mathbb{P}_\lambda} q$ and $x \notin a_{p_0}$. If we define

$$p \ = \ \langle s_{p_0}, t_{p_0}, \vec{d}_{p_0}, \{\langle x, \emptyset \rangle\} \cup \langle c_{p_0,y} \mid y \in a_{p_0} \rangle, \vec{A}_{p_0} \rangle,$$

then the above assumptions imply that p is a condition in $\mathbb{P}_\lambda$ that is stronger than p_0. Hence the set of all conditions p in $\mathbb{P}_\lambda$ with $x \in a_p$ is dense below $q \in G$.

The second statement of the claim is immediate from its first statement and the definition of $\mathbb{P}_\lambda$. □

Proposition 6.4. *If G is $\mathbb{P}_\lambda$-generic over V and $x \in D(G)$, then*

$$\kappa \ = \ \sup\{\sup(c_{p,x}) \mid p \in G, \ x \in a_p\}$$

and for every $\alpha < \kappa$, either $\kappa = \sup\{\sup(d_{p,\alpha}) \mid p \in G\}$ or $d_{p,\alpha} = \emptyset$ for every $p \in G$ with $\alpha \leq \beta_p$. Moreover the latter case occurs if and only if there is a condition $p \in G$ such that $\alpha \leq \beta_p$ and $d_{p,\alpha}$ is required to be the empty set by one of the statements listed in Clause (iii) of Definition 5.2.

Proof. Fix a condition q in $\mathbb{P}_\lambda$ with $x \in a_q$ and $\beta \in S$ with $\beta_q < \beta < \kappa$. Moreover, if there is an $\alpha \leq \beta_q$ such that $d_{q,\alpha}$ is not required to be the empty set by one of the statements listed in Clause (iii) of Definition 5.2, then we also fix such an ordinal α. We define p to be the tuple

$$\langle s, t, \langle d_\zeta \mid \zeta \leq \beta \rangle, \langle c_x \mid x \in a_q \rangle, \vec{A}_q \rangle$$

with

- $s = s_q \cup \langle \langle \xi, \emptyset \rangle \mid \beta_q < \xi \leq \beta \rangle$.
- $t = t_q \cup \langle \langle \xi, 1 \rangle \mid \beta_q < \xi \leq \beta \rangle$.
- $c_x = c_{q,x} \cup (\beta_q, \beta]$ for all $x \in a_q$.
- $d_\zeta = \begin{cases} d_{q,\zeta}, & \text{if } \zeta \leq \beta_q \text{ with } \alpha \neq \zeta. \\ d_{q,\alpha} \cup \{\beta\}, & \text{if } \alpha = \zeta. \\ \emptyset, & \text{if } \beta_q < \zeta \leq \beta. \end{cases}$

Then p is a condition in $\mathbb{P}_\lambda$ with $p \leq_{\mathbb{P}_\lambda} q$, $\beta_p = \beta$, $\sup(c_{p,x}) = \beta$ and $\sup(d_{p,\alpha}) = \beta$. Together with Proposition 6.3, this implies all but the backwards direction of the last statement of the claim. To see that this direction holds as well, note that if $p \in G$ and $d_{p,\alpha}$ is required to be the empty set by one of the statements in Clause (iii) of Definition 5.2 and q is a condition in $\mathbb{P}_\lambda$ below p, then $d_{q,\alpha}$ is required to be the empty set by that same statement and hence $d_{q,\alpha} = \emptyset$ for every $q \in G$ with $\alpha \leq \beta_q$. □

We fix $\mathbb{P}_\lambda$-names $\dot{s}$ and $\dot{t}$ in V such that $\dot{s}^H = \bigcup\{s_p \mid p \in H\} : \kappa \longrightarrow {}^{<\kappa}2$ and $\dot{t}^H = \bigcup\{t_p \mid p \in H\} : \kappa \longrightarrow 2$ holds whenever H is $\mathbb{P}_\lambda$-generic over V.

Lemma 6.5. *If G is $\mathbb{P}_\lambda$-generic over* V*, then $D(G)$ is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameters $\dot{s}^G$ and $\dot{t}^G$.*

Proof. Let G be $\mathbb{P}_\lambda$-generic over V. The statement of the lemma will be a consequence of the following two claims. The first claim is a direct consequence of the definition of $\mathbb{P}_\lambda$ and Proposition 6.4.

Claim. *If $x \in D(G)$, then $C^x_G = \bigcup\{c_{p,x} \mid p \in G,\ x \in a_p\}$ is a club subset of κ such that the implication*

$$\dot{s}^G(\alpha) \subseteq x \ \longrightarrow\ \dot{t}^G(\alpha) = 1 \tag{2}$$

holds for all $\alpha \in C^x_G$. □

Claim. *Assume that $x \in ({}^\kappa 2)^{\mathrm{V}[G]}$ is such that the implication (2) holds for every element α of some club subset C of κ. Then x is an element of $D(G)$.*

Proof of the Claim. Note that the following proof will be similar to the proof of Lemma 4.1. Let $\dot{a}$ be the canonical $\mathbb{P}_\lambda$-name such that $\dot{a}^H = \bigcup\{a_p \mid p \in H\}$ holds whenever H is $\mathbb{P}_\lambda$-generic over V. Assume, towards a contradiction, that x is not an element of $\dot{a}^G = D(G)$. Then we can find $q \in G$ and $\mathbb{P}_\lambda$-names $\dot{C}$ and $\dot{x}$ such that $x = \dot{x}^G$ and

$$q \Vdash_{\mathbb{P}_\lambda} \text{“}\dot{x} \in {}^{\check{\kappa}}2 \setminus \dot{a} \ \wedge \ \dot{C} \subseteq \check{\kappa} \textit{ club} \ \wedge \ \forall \alpha \in \dot{C} \ [\dot{s}(\alpha) \subseteq \dot{x} \longrightarrow \dot{t}(\alpha) = 1]\text{”}$$

Fix a condition p_0 in $\mathbb{P}_\lambda$ below q. Let M be a countable elementary substructure of $\langle \mathrm{H}(\theta), \in \rangle$ for some large, regular θ with the property that $\beta = \sup(M \cap \kappa) \in S^*$ and $\mathbb{P}_\lambda, p_0, q, \dot{a}, \dot{C}, \dot{x} \in M$. Pick a decreasing sequence of conditions $\langle p_n \mid n < \omega \rangle \subseteq M$ so that $p_n \in D$ for some n whenever $D \in M$ is a dense subset of $\mathbb{P}_\lambda$. By the genericity of the p_n and the fact that forcing with $\mathbb{P}_\lambda$ preserves the regularity of κ, we have $\beta = \sup_{n<\omega} \beta_{p_n}$ and (using Lemma 3.2 and Lemma 5.9) there is $u : \beta \longrightarrow 2$ such that for every $n < \omega$ there is an $m \geq n$ with $p_m \Vdash_{\mathbb{P}_\lambda}$ “$\dot{x} \restriction \check{\beta}_{p_n} = \check{u} \restriction \check{\beta}_{p_n}$” and $y \restriction \beta_{p_n} \neq u \restriction \beta_{p_n}$ for all $y \in a_{p_n}$. Define

$$p \ = \ \langle s, t, \langle d_\alpha \mid \alpha \leq \beta \rangle, \langle c_y \mid y \in a \rangle, \vec{A} \rangle$$

by setting

- $s \ = \ \{\langle \beta, u \rangle\} \ \cup \ \bigcup_{n<\omega} s_{p_n}$.
- $t \ = \ \{\langle \beta, 0 \rangle\} \ \cup \ \bigcup_{n<\omega} t_{p_n}$.
- $\bar{d}_\alpha \ = \ \bigcup \{d_{p_n,\alpha} \mid n < \omega, \ \alpha \leq \beta_{p_n}\}$ for every $\alpha < \beta$.
- $d_\alpha \ = \ \begin{cases} \bar{d}_\alpha \cup \{\beta\}, & \text{if } \alpha < \beta \text{ and } \bar{d}_\alpha \neq \emptyset. \\ \emptyset, & \text{if either } \alpha = \beta \text{ or } \alpha < \beta \text{ and } \bar{d}_\alpha = \emptyset. \end{cases}$
- $a \ = \ \bigcup_{n<\omega} a_{p_n}$.
- $c_y \ = \ \{\beta\} \ \cup \ \bigcup \{c_{p_\alpha,y} \mid \alpha < \eta, \ y \in a_{p_\alpha}\}$ for every $y \in a$.
- $\vec{A} \ = \ \bigcup_{n<\omega} \vec{A}_{p_n}$.

Since $\beta \in S^*$ and $u \nsubseteq y$ for every $y \in a$, we can conclude that p is a condition in $\mathbb{P}_\lambda$ that is stronger than p_0. This construction ensures

$$p \Vdash_{\mathbb{P}_\lambda} \text{“}\check{\beta} \in \dot{C} \ \wedge \ \dot{s}(\check{\beta}) = \check{s}(\check{\beta}) \subseteq \dot{x} \ \wedge \ \dot{t}(\check{\beta}) = 0\text{”},$$

a contradiction. Hence we can conclude that $x \in \dot{a}^G$. □

By the above claims we can conclude that

$$D(G) \ = \ \{x \in ({}^\kappa 2)^{\mathrm{V}[G]} \mid \exists C \subseteq \kappa \textit{ club} \ \forall \alpha \in C \ [\dot{s}^G(\alpha) \subseteq x \ \longrightarrow \ \dot{t}^G(\alpha) = 1]\}.$$

This yields a Σ_1-definition of $D(G)$ over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ using the parameters $\dot{s}^G$ and $\dot{t}^G$. □

Lemma 6.6. *Let $\alpha < \kappa$ and G is $\mathbb{P}_\lambda$-generic over* V*. Then S_α is a stationary subset of κ in* $\mathrm{V}[G]$ *if and only if there is a $p \in G$ such that $d_{p,\alpha}$ is required to be the empty set by one of the statements listed in Clause (iii) of Definition 5.2.*

Proof. By Lemma 5.9, the partial order of conditions in $\mathbb{P}_\lambda$ below p is strongly S^*_α-complete whenever $d_{p,\alpha}$ is required to be the empty set by one of the statements listed in Clause (iii) of Definition 5.2. Thus, if there is such a condition p in G, then Corollary 3.6 shows that forcing with $\mathbb{P}_\lambda$ preserves the fat stationarity of S^*_α. In the other case, forcing with $\mathbb{P}_\lambda$ destroys the stationarity of S_α, because Proposition 6.4 shows that $\bigcup\{d_{p,\alpha} \mid p \in G, \ \alpha \leq \beta_p\}$ is a closed unbounded subset of $\tilde{S}_\alpha$ in $\mathrm{V}[G]$. □

Lemma 6.7. *Let G be $\mathbb{P}_\lambda$-generic over V. Then the sets $\dot{s}^G$, $\dot{t}^G$ and $\vec{T}$ are Δ_1-definable in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ using the sequence $\vec{S}$ as a parameter.*

Proof. Using Clause (iii) in Definition 5.2 and Lemma 6.6, it follows that the following equivalences hold for all $\beta, \gamma, \delta < \kappa$ with $\beta = \prec\gamma, \delta\succ$.

$$\begin{aligned}
\dot{s}^G(\gamma)(\delta) = 0 &\iff S_{\beta\cdot 7} \text{ is stationary} \\
&\iff S_{\beta\cdot 7+1} \text{ is not stationary } \wedge \ S_{\beta\cdot 7+2} \text{ is not stationary.} \\
\dot{s}^G(\gamma)(\delta) = 1 &\iff S_{\beta\cdot 7+1} \text{ is stationary} \\
&\iff S_{\beta\cdot 7} \text{ is not stationary } \wedge \ S_{\beta\cdot 7+2} \text{ is not stationary.} \\
\dot{t}^G(\beta) = 0 &\iff S_{\beta\cdot 7+3} \text{ is stationary } \iff S_{\beta\cdot 7+4} \text{ is not stationary.} \\
\dot{t}^G(\beta) = 1 &\iff S_{\beta\cdot 7+4} \text{ is stationary } \iff S_{\beta\cdot 7+3} \text{ is not stationary.} \\
\dot{s}^G(\gamma) \notin T_\delta &\iff S_{\beta\cdot 7+5} \text{ is stationary } \iff S_{\beta\cdot 7+6} \text{ is not stationary.} \\
\dot{s}^G(\gamma) \in T_\delta &\iff S_{\beta\cdot 7+6} \text{ is stationary } \iff S_{\beta\cdot 7+5} \text{ is not stationary.}
\end{aligned}$$

These equivalences yield Δ_1-definitions of $\dot{s}^G$, $\dot{t}^G$ and $\vec{T}$ in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that only use the sequence $\vec{S}$ as a parameter. □

Lemma 6.8. *Let G be $\mathbb{P}_\lambda$-generic over V. Then there is a well-order of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameter $\vec{S}$.*

Proof. Define $W = \{w_\delta \mid \delta < \lambda\}$. Then our assumptions (made at the beginning of Section 5) imply that W and $\vec{w}$ are both definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by Σ_1-formulas that use the sequence $\vec{T}$ of subtrees of ${}^{<\kappa}\kappa$ as a parameter.

Claim. *If $p \in G$ and $\delta < \gamma_p$, then*

$$\dot{A}^G_{p,\delta} \ = \ \{\alpha < \kappa \mid w_\delta \oplus c(\alpha, 1) \in D(G)\} \ = \ \{\alpha < \kappa \mid w_\delta \oplus c(\alpha, 0) \notin D(G)\}.$$

Proof of the Claim. By the definition of $D(G)$, we have

$$\alpha \in \dot{A}^G_{p,\delta} \iff \exists q \in G \ [\delta < \gamma_q \wedge \alpha \in \dot{A}^G_{q,\delta}] \iff w_\delta \oplus c(\alpha, 1) \in D(G)$$

and

$$\alpha \notin \dot{A}^G_{p,\delta} \iff \exists q \in G \ [\delta < \gamma_q \wedge \alpha \notin \dot{A}^G_{q,\delta}] \iff w_\delta \oplus c(\alpha, 0) \in D(G). \quad \square$$

Working in $\mathrm{V}[G]$, we define P to be the set of all pairs $\langle z, v\rangle$ such that $z \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, $v \in W$ and there is a subset A of κ coding z that satisfies

$$[\alpha \in A \longrightarrow v \oplus c(\alpha, 1) \in D(G)] \ \wedge \ [\alpha \notin A \longrightarrow v \oplus c(\alpha, 0) \in D(G)].$$

Claim. *Let $z \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and let δ_z be the unique ordinal (given by Lemma 6.1) such that $\delta_z < \gamma_p$ and $\dot{A}^G_{p,\delta_z}$ codes z for some $p \in G$. Then w_{δ_z} is the unique element of W with $\langle z, w_{\delta_z}\rangle \in P$.*

Proof of the Claim. By the previous claim, the subset $\dot{A}^G_{p,\delta_z}$ of κ witnesses that $\langle z, w_{\delta_z}\rangle$ is an element of P. Now assume, towards a contradiction, that there is $\delta < \lambda$ with $\delta \neq \delta_z$ and $\langle z, w_\delta\rangle \in P$. Let $A \subseteq \kappa$ witness that $\langle z, w_\delta\rangle \in P$. By the above claim, $A = \dot{A}^G_{q,\delta}$ for some $q \in G$ with $\bar{\gamma} = \max\{\delta, \delta_z\} < \gamma_q$. If we set $\bar{G} = G \cap \mathbb{P}_{\bar{\gamma}}$, then Corollary 5.12 implies $|\bar{\gamma}|^{\mathrm{V}[\bar{G}]} < |\lambda|^{\mathrm{V}[\bar{G}]}$ and the subsets $\dot{A}^{\bar{G}}_{q,\delta} = \dot{A}^G_{q,\delta}$ and $\dot{A}^{\bar{G}}_{q,\delta_z} = \dot{A}^G_{q,\delta_z}$ code the same element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[\bar{G}]}$. This contradicts Clause (v) of Definition 5.1. □

Let $\prec_{\vec{w}}$ denote the wellorder on W induced by its enumeration $\vec{w}$. Define $\prec_*$ to be the set of all pairs $\langle z, \bar{z}\rangle$ in $\mathrm{H}(\kappa^+)$ such that

$$\exists v, \bar{v} \in W \ [\langle z, v\rangle \in P \ \wedge \ \langle \bar{z}, \bar{v}\rangle \in P \ \wedge \ v \prec_{\vec{w}} \bar{v}].$$

Lemma 6.5 implies that P is Σ_1-definable over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ using parameters $\vec{T}$, $\dot{s}^G$ and $\dot{t}^G$. Thus the assumptions made at the beginning of Section 5 imply that $\prec_*$ is Σ_1-definable over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ using parameters $\vec{T}$, $\dot{s}^G$ and $\dot{t}^G$. Lemma 6.7 shows that each of these parameters is itself definable in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ by a Σ_1-formula with parameter $\vec{S}$. In particular, the relation $\prec_*$ is definable over $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ by a Σ_1-formula with parameter $\vec{S}$.

Given $z_0, z_1 \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and $\delta_0, \delta_1 < \lambda$ such that δ_i is the unique ordinal with the property that $\delta_i < \gamma_p$ and $\dot{A}^G_{p,\delta_i}$ codes z_i for some $p \in G$, we have $z_0 \prec_* z_1$ if and only if $\delta_0 < \delta_1$. This shows that $\prec_*$ is a well-order of $\mathrm{H}(\kappa^+)$. □

Proof of Theorem 1.4. Let κ and $\langle S_\alpha \mid \alpha \leq \kappa\rangle$ be as in the statement of the theorem and $\lambda = 2^\kappa$. Fix an injective sequence $\vec{w} = \langle w_\gamma \mid \gamma < \lambda\rangle$ of elements of ${}^\kappa 2$ and define $A = \{w_\delta \oplus w_\gamma \mid \delta < \gamma < \lambda\}$. Fix an enumeration $\vec{s}$ of ${}^{<\kappa}\kappa$ as in Definition 2.1. Let $C_{\vec{s}}(A)$ be the notion of forcing corresponding to A given by Definition 2.1. Since forcing with $C_{\vec{s}}(A)$ preserves our assumptions on κ and Corollary 4.2 shows that all assumptions listed at the beginning of Section 5 hold in $C_{\vec{s}}(A)$-generic extensions of the ground model V, there is a canonical $C_{\vec{s}}(A)$-name $\dot{\mathbb{Q}}$ with the property that $\dot{\mathbb{Q}}^G = \mathbb{P}_\lambda$ whenever G is $C_{\vec{s}}(A)$-generic over V and $\mathbb{P}^{\mathrm{V}[G]}_{\vec{w}} = \langle \mathbb{P}_\gamma \mid \gamma \leq \lambda\rangle$ is the corresponding sequence of partial orders constructed in $\mathrm{V}[G]$ with respect to $\vec{w}$.

Then the combination of Proposition 2.2, Lemma 5.9, Corollary 5.12 and Corollary 6.2 implies that $\mathbb{P} = C_{\vec{s}}(A) * \dot{\mathbb{Q}}$ is $<\kappa$-distributive and forcing with $\mathbb{P}$ preserves all cofinalities less than or equal to λ and the value of 2^κ. If $G * H$ be $(C_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic over V, then Lemma 6.8 implies that there is a well-order of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G*H]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G*H]}, \in\rangle$ by a Σ_1-formula with parameter $\vec{S}$. □

7. Σ_1-definable Sequences of disjoint fat stationary Sets

This section contains the proofs of Theorem 1.5 and Theorem 1.7. We start with a definition that will allow us to prove both theorems using the same techniques.

Definition 7.1. Let κ be an uncountable regular cardinal. We say that a tuple $\langle \delta, \theta, \nu, \lhd\rangle$ is *suitable for* κ if $\theta > \kappa$ is a regular cardinal, $\delta \geq \theta$ is a strong limit cardinal, $\nu \leq \kappa$ is an ordinal, $\lhd$ is a well-ordering of $\mathrm{H}(\theta)$ of order-type θ and the following statements hold.

(i) The set $I(\lhd) = \{\{x \mid x \lhd y\} \mid y \in \mathrm{H}(\theta)\}$ of all proper initial segments of $\lhd$ is definable over $\langle \mathrm{H}(\theta), \in\rangle$ by a Σ_1-formula with parameter ν.

(ii) If $\mathbb{P}$ is a partial order of cardinality less than δ with the property that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and G is $\mathbb{P}$-generic over V, then $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν.

(iii) There is a closed unbounded subset of $[\mathrm{H}(\theta)]^{<\kappa}$ consisting of elementary submodels M of $\mathrm{H}(\theta)$ with $\pi[I(\lhd) \cap M] \subseteq I(\lhd)$, where $\pi : M \longrightarrow N$ denotes the corresponding transitive collapse.

The following proposition shows that the last statement listed in the above definition follows from the first statement in the case where $\nu < \kappa$.

Proposition 7.2. *Let $\kappa < \theta$ be uncountable regular cardinals and let $\lhd$ be a well-ordering of $\mathrm{H}(\theta)$. Assume that there is $\nu < \kappa$ such that the set $I(\lhd)$ of all proper initial segments of $\lhd$ is definable over $\langle \mathrm{H}(\theta), \in\rangle$ by a Σ_1-formula with parameter ν. If M is an elementary substructure of $\mathrm{H}(\theta)$ of cardinality less than κ with $\kappa, \nu \in M$ and $M \cap \kappa \in \kappa$ and $\pi : M \longrightarrow N$ denotes the corresponding transitive collapse, then $\pi[I(\lhd) \cap M] \subseteq I(\lhd)$ holds.*

Proof. Fix a Σ_0-formula $\varphi(v_0, v_1, v_2)$ such that

$$I(\lhd) \;=\; \{A \in \mathrm{H}(\theta) \mid \exists x \in \mathrm{H}(\theta)\ \varphi(A, x, \nu)\}$$

and pick $A \in I(\lhd) \cap M$. By elementarity, there is $X \in M$ such that $\varphi(A, X, \nu)$ holds. Since $\pi(\nu) = \nu$ and $N \subseteq \mathrm{H}(\theta)$, we can apply Σ_0-absoluteness to conclude that $\varphi(\pi(A), \pi(X), \nu)$ holds and $\pi(A)$ is an element of $I(\lhd)$. □

In the following, we present two settings in which the above requirements are satisfied. We start by showing that in L there is a suitable tuple for every uncountable regular cardinal.

Lemma 7.3. *Assume that* $\mathrm{V} = \mathrm{L}$ *holds. If* $\theta > \kappa$ *is a regular cardinal,* $\delta \geq \theta$ *is a strong limit cardinal and* $\lhd$ *denotes the restriction of the canonical well-ordering of* L *to* $\mathrm{H}(\theta)$*, then the tuple* $\langle \delta, \theta, 0, \lhd \rangle$ *is suitable for* κ*.*

Proof. The set $I(\lhd)$ consists of all $A \in \mathrm{H}(\theta)$ with the property that there is an $\alpha < \theta$ with $A \in \mathrm{L}_\alpha$ and $\langle \mathrm{L}_\alpha, \in \rangle \models$ "*A is an initial segment of* $<_{\mathrm{L}}$". This shows that $I(\lhd)$ is definable over $\langle \mathrm{H}(\theta), \in \rangle$ by a Σ_1-formula without parameters.

Let $\mathbb{P}$ be a partial order with the property that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and let G be $\mathbb{P}$-generic over V. Then $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is equal to $\mathrm{L}_{(\kappa^+)^{\mathrm{V}[G]}}$. This shows that the set $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula without parameters. □

Next, we present a setting in which suitable tuples exists for all uncountable regular cardinals below a measurable cardinal. The following arguments make use of some basic properties of the *Dodd-Jensen core model* K^{DJ} (see [4]). The following observation will allow us to show that the third clause of Definition 7.1 is satisfied in this setting.

Proposition 7.4. *Assume that* $\mathrm{V} = \mathrm{L}[E]$ *is an extender model in the sense of* [21]*. If* κ *is an uncountable regular cardinal,* $\lhd$ *is the restriction of the canonical well-ordering of* $\mathrm{L}[E]$ *to* $\mathrm{H}(\kappa^+)$ *and* $I(\lhd)$ *is the set of all proper initial segments of* $\lhd$*, then there is a closed unbounded subset of* $[\mathrm{H}(\kappa^+)]^{<\kappa}$ *consisting of elementary submodels* M *of* $\mathrm{H}(\kappa^+)$ *with* $\pi[I(\lhd) \cap M] \subseteq I(\lhd)$*, where* $\pi : M \longrightarrow N$ *denotes the corresponding transitive collapse.*

Proof. By [6, Theorem 8], we know that *Local Club Condensation* holds and we can use [7, Theorem 88] to find a closed unbounded subset of $[\mathrm{H}(\kappa^+)]^{<\kappa}$ consisting of elementary submodels of $\mathrm{H}(\kappa^+)$ with the desired properties. □

In the proof of the following lemma, we use the presentation of the core model given in [13]. In the proof of the lemma, we only consider premice over the empty set. Therefore we omit the index D (as used in [13]) in these arguments.

Lemma 7.5. *Assume that U is a normal measure on a cardinal δ and $\mathrm{V} = \mathrm{L}[U]$ holds. If $\kappa < \delta$ is an uncountable regular cardinal and $\lhd$ denotes the restriction of the canonical well-ordering of the Dodd-Jensen core model K^{DJ} to $\mathrm{H}(\kappa^+)^{\mathrm{K}^{DJ}}$, then the tuple $\langle\delta, \kappa^+, \kappa, \lhd\rangle$ is suitable for κ.*

Proof. Let K denote the Dodd-Jensen core model K^{DJ}. Then the results of [4] show that $\mathrm{H}(\delta) \subseteq \mathrm{K}$ and $\lhd$ is a well-ordering of $\mathrm{H}(\kappa^+)$ of order-type κ^+. Let $\mathbb{P} \in \mathrm{H}(\delta)$ be a (possibly trivial) partial order with the property that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and let G be $\mathbb{P}$-generic over V. Then we have $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]} = \mathrm{H}(\kappa^+)^{\mathrm{K}[G]}$ and the results of [13] show that $\mathrm{K} = (\mathrm{K}^{DJ})^{\mathrm{K}[G]}$. By [13, Theorem 2.7], the set of all *mice* in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameter κ. Since $\mathrm{V} = \mathrm{L}[U]$ and $\mathrm{H}(\delta)^{\mathrm{V}} \subseteq \mathrm{K}$, a standard argument using elementary substructures of $\mathrm{H}(\delta^+)^{\mathrm{V}}$ of cardinality κ shows that $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is equal to the union of all *low parts* $\mathrm{lp}(M)$ of mice M (see [13, Section 1]) in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and we can conclude that $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameter κ. Moreover, given $A \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, we can use [13, Theorem 2.10] and [13, Theorem 3.4] to see that A is an element of $I(\lhd)$ if and only if there is a mouse $M = \mathrm{J}_\alpha[F] \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ such that $A \in \mathrm{lp}(M)$ and $\langle M, \in\rangle \models$ "*A is a proper initial segment of $<_{\mathrm{J}[F]}$*". This shows that $I(\lhd)$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameter κ. Finally, K is an extender model and therefore Proposition 7.4 implies that the third clause of Definition 7.1 is satisfied in this setting. □

In the subsequent paper mentioned after Question 1.9 in Section 1, we will show that if M_1 exists, δ is the unique Woodin cardinal in M_1 and $\lhd$ is the canonical well-ordering of $\mathrm{H}(\omega_2)$ in M_1, then the tuple $\langle\delta, \omega_2, \omega_1, \lhd\rangle$ is suitable for ω_1 in M_1. With the help of the techniques developed in this chapter, we will use this result in that paper to show that the existence of a well-ordering of $\mathrm{H}(\omega_2)$ that is locally definable by a Σ_1-formula with parameter ω_1 is consistent with the existence of a Woodin cardinal and a failure of the GCH at ω_1.

Next, we show how the concept of κ-suitable tuples can be combined with our previous forcing constructions to obtain well-orders of $\mathrm{H}(\kappa^+)$ whose Σ_1-definition only uses the cardinal κ as a parameter. We start by proving some direct consequences of suitability.

Lemma 7.6. *Let κ be an uncountable regular cardinal and let $\langle\delta,\theta,\nu,\lhd\rangle$ be suitable for κ. Define $\lhd_\kappa = \lhd \upharpoonright (\mathrm{H}(\kappa)\times\mathrm{H}(\kappa))$.*

- *(i) The order $\lhd$ is definable over $\langle\mathrm{H}(\theta),\in\rangle$ by a Σ_1-formula with parameter ν.*
- *(ii) The set $\mathrm{H}(\kappa)$ is an element of $I(\lhd)$ and κ is $\lhd_\kappa$-cofinal in $\mathrm{H}(\kappa)$.*
- *(iii) The well-order $\langle\mathrm{H}(\kappa),\lhd_\kappa\rangle$ has order-type κ and $\kappa=\kappa^{<\kappa}$ holds.*
- *(iv) The set $\{\mathrm{H}(\kappa)\}$ is definable over $\langle\mathrm{H}(\theta),\in\rangle$ by a Σ_1-formula with parameters κ and ν.*
- *(v) The set $\{\lhd_\kappa\}$ is definable over $\langle\mathrm{H}(\theta),\in\rangle$ by a Σ_1-formula with parameters κ and ν.*
- *(vi) If κ is the successor of a regular cardinal η, then the set $\{S^\kappa_\eta\}$ is definable over $\langle\mathrm{H}(\theta),\in\rangle$ by a Σ_1-formula with parameters κ and ν.*

Proof. (i) Given $x,y\in\mathrm{H}(\theta)$, we have $x\lhd y$ if and only if there are $A,B\in I(\lhd)$ with $x\in A$ and $y\in B\setminus A$. By our assumptions on $I(\lhd)$, this yields the first statement of the lemma.

(ii) Fix $x,y\in\mathrm{H}(\theta)$ with $x\lhd y$ and $y\in\mathrm{H}(\kappa)$. Since $\lhd$ has order-type θ, there is $\lambda<\theta$ with $y\lhd\lambda$. Pick $A,B\in I(\lhd)$ with $y\in A$ and $\lambda\in B\setminus A$. Choose an elementary submodel M of $\mathrm{H}(\theta)$ contained in the closed unbounded set described in Definition 7.1.(iii) with $\mathrm{tc}(\{y\})\cup\{A,B,\lambda,\kappa,\nu\}\subseteq M$ and let $\pi : M\longrightarrow N$ denote the corresponding transitive collapse. Then we have $\pi(A),\pi(B)\in I(\lhd)$, $y=\pi(y)\in\pi(A)\in I(\lhd)$ and hence $x\in\pi(A)\subseteq N\subseteq\mathrm{H}(\kappa)$. Moreover, we have $y=\pi(y)\in\pi(B)\setminus\pi(A)$, $\pi(\lambda)\in\pi(B)$ and hence $y\lhd\pi(\lambda)<\kappa$.

(iii) The second statement of the lemma implies that the well-order $\langle\mathrm{H}(\kappa),\lhd_\kappa\rangle$ has order-type at least κ. Assume toward a contradiction that there is $A\in I(\lhd)$, a bijection $b:\kappa\longrightarrow A$ and $y\in\mathrm{H}(\kappa)$ with $A=\{x\in\mathrm{H}(\kappa)\mid x\lhd y\}$. Choose an elementary submodel M of $\mathrm{H}(\theta)$ contained in the closed unbounded set described in Definition 7.1.(iii) with $\mathrm{tc}(\{y\})\cup\{A,b,\kappa,\nu\}\subseteq M$ and let $\pi : M\longrightarrow N$ denote the corresponding transitive collapse. Then we have $\pi(A),\pi(A\cup\{y\})\in I(\lhd)$, $\pi(A)\cup\{y\}=\pi(A\cup\{y\})$, which shows that $A=\pi(A)$. But elementarity implies that there is a bijection between $\pi(A)$ and $\pi(\kappa)<\kappa$, a contradiction. In particular, this shows that $\mathrm{H}(\kappa)$ has cardinality κ and hence we also get $\kappa=\kappa^{<\kappa}$.

(iv) By the second part of the lemma, $\mathrm{H}(\kappa)$ is the unique element M of $\mathrm{H}(\theta)$ such that $M\in I(\lhd)$, $M\cap\kappa=\kappa$ and κ is $\lhd$-cofinal in M. By Definition 7.1.(i) and the first part of the lemma, this yields the statement of the claim.

(v) This statement follows from our assumptions and the above statements.

(vi) Since the sets S^κ_η and $\{\eta\}$ are definable over $\langle \mathrm{H}(\kappa), \in\rangle$, this statement follows directly from the fourth statement of the lemma. □

The next result shows how we can use suitable tuples to replace sequences of fat stationary subsets of κ by the cardinal κ as the parameter in a Σ_1-definition of a well-ordering of $\mathrm{H}(\kappa^+)$.

Theorem 7.7. *Let κ be an uncountable regular cardinal and let $\langle \delta, \theta, \nu, \lhd\rangle$ be suitable for κ. Then there is a sequence $\langle S_\alpha \mid \alpha \leq \kappa\rangle$ of disjoint fat stationary subsets of κ with the property that the set $\{\langle S_\alpha \mid \alpha < \kappa\rangle\}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν whenever $\mathbb{P}$ is a partial order of cardinality less than δ such that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and G is $\mathbb{P}$-generic over* V*.*

Proof. Define $F : \kappa \cap \mathrm{Lim} \longrightarrow \kappa$ to be the unique function such that the following statements hold for all $\alpha \in \kappa \cap \mathrm{Lim}$.

- (i) Assume that there is a triple $\langle \gamma, \lambda, C\rangle$ such that $\gamma, \lambda < \alpha$ and $C \subseteq \alpha \cap \mathrm{Lim}$ is a club in α with the property that, for every closed bounded subset c of C of order-type λ, there is $\bar{\alpha} \in c$ with $F(\bar{\alpha}) \neq \gamma$. Then $F(\alpha) = \gamma_0$, where $\langle \gamma_0, \lambda_0, C_0\rangle$ denotes the $\lhd$-minimal triple with this property.
- (ii) If there is no such triple, then $F(\alpha) = \alpha$.

By the Recursion Theorem, F is definable over $\langle \mathrm{H}(\theta), \in\rangle$ by a formula with parameters κ and ν.

Claim. *Given $\beta < \kappa$, the set $F^{-1}\{\beta\}$ is a fat stationary subset of κ.*

Proof. Assume, towards a contradiction, that the statement of the claim fails. Then there is a triple $\langle \beta, \lambda, C\rangle$ such that $\beta, \lambda < \kappa$ and $C \subseteq \kappa \cap \mathrm{Lim}$ is a club subset of κ with the property that, for every closed bounded subset c of C of order-type λ, there is an $\alpha \in c$ with $F(\alpha) \neq \beta$. Let $\langle \beta_0, \lambda_0, C_0\rangle$ denote the $\lhd$-minimal triple with these properties and set

$$A \;=\; \{x \mid x \lhd \langle \beta_0, \lambda_0, C_0\rangle\} \;\cup\; \{\langle \beta_0, \lambda_0, C_0\rangle\} \;\in\; I(\lhd).$$

Since $\langle \delta, \theta, \nu, \lhd\rangle$ is suitable for κ, we can find a monotone enumeration $\langle \alpha_\xi \mid \xi < \kappa\rangle$ of a club subset of κ and a continuous ascending sequence $\langle M_\xi \mid \xi < \kappa\rangle$ of elementary submodels of $\mathrm{H}(\theta)$ of size less than κ such that the following statements hold for all $\xi < \kappa$.

- (1) $\alpha_\xi = \kappa \cap M_\xi$ and $\beta_0, \lambda_0, \kappa, \nu, A, C_0 \in M_\xi$.
- (2) M_ξ is contained in the club described in Definition 7.1.(iii), thus if $\pi_\xi : M_\xi \longrightarrow N_\xi$ denotes the transitive collapse of M_ξ, then we have $\pi[I(\lhd) \cap M] \subseteq I(\lhd)$.

Fix $\xi < \kappa$. Then $\beta_0, \lambda_0 < \alpha_\xi = \pi_\xi(\kappa) \in \mathrm{Lim}$, $C_0 \cap \alpha_\xi = \pi_\xi(C_0) \subseteq \alpha_\xi \cap \mathrm{Lim}$ is a club in α_ξ and $\alpha_\xi \in C_0$. Pick a closed bounded subset c of $C_0 \cap \alpha_\xi$ of order-type λ. Then c is a closed bounded subset of C_0 and there is $\bar{\alpha} \in c$ with $F(\bar{\alpha}) \neq \beta_0$. This shows that the triple $\langle \beta_0, \lambda_0, C_0 \cap \kappa_\alpha \rangle$ satisfies the assumption in (i) with respect to α_ξ and hence $F(\alpha_\xi) < \alpha_\xi$ by elementarity of M_ξ. Next, if such exists, pick $\langle \gamma, \rho, D \rangle \lhd \langle \beta_0, \lambda_0, C_0 \cap \kappa_\alpha \rangle$ such that $\gamma, \rho < \alpha_\xi$ and $D \subseteq \alpha_\xi \cap \mathrm{Lim}$ is a club in α_ξ. Since (2) implies that $\pi_\xi(A) \in I(\lhd)$ and $\langle \gamma, \rho, D \rangle \in \pi_\xi(A) \subseteq N_\xi$, there is $\bar{D} \subseteq \kappa \cap \mathrm{Lim}$ club in κ such that $\bar{D} \in M_\xi$, $D = \pi_\xi(\bar{D}) = \bar{D} \cap \alpha_\xi$ and $\langle \gamma, \rho, D \rangle \lhd \langle \beta_0, \lambda_0, C_0 \rangle$. In this situation, the $\lhd$-minimality of $\langle \beta_0, \gamma_0, C_0 \rangle$ and elementarity implies that there is a closed bounded subset d of $\bar{D}$ of order-type ρ such that $d \in M_\xi$ and $F(\alpha) = \gamma$ for all $\alpha \in d$. Then $\sup(d) < \alpha_\xi$ and d is a closed bounded subset of D of order-type ρ. These computations show that $\langle \beta_0, \lambda_0, C_0 \cap \kappa_\alpha \rangle$ is the $\lhd$-minimal triple that satisfies the assumption in (i) with respect to α_ξ, and we can conclude that $F(\alpha_\xi) = \beta_0$.

Set $c = \{\alpha_\xi \mid \xi < \lambda_0\}$. We have just shown that c is a closed bounded subset of C_0 with $F(\alpha) = \beta_0$ for all $\alpha \in c$. This contradicts the choice of $\langle \beta_0, \lambda_0, C_0 \rangle$. □

Set $S_\kappa = F^{-1}\{0\}$ and $S_\beta = F^{-1}\{1 + \beta\}$ for all $\beta < \kappa$. By the above claim, the sequence $\langle S_\beta \mid \beta \leq \kappa \rangle$ consists of pairwise disjoint fat stationary subsets of κ.

Claim. *The set $\{\langle S_\beta \mid \beta < \kappa \rangle\}$ is definable over $\langle \mathrm{H}(\theta), \in \rangle$ by a Σ_1-formula with parameters κ and ν.*

Proof of the Claim. By Lemma 7.6, the set $\{\langle \mathrm{H}(\kappa), \in, \lhd_\kappa \rangle\}$ is definable over the structure $\langle \mathrm{H}(\theta), \in \rangle$ by a Σ_1-formula with parameters κ and ν. Since $\langle \mathrm{H}(\kappa), \in, \lhd_\kappa \rangle$ is a model of $\mathrm{ZFC}^-_{\dot{A}}$ (the canonical extension of the axioms of ZFC^- to the language of set theory extended by a new predicate symbol $\dot{A}$, that includes all instances of Replacement and Separation for formulas in the extended language), we can use the Recursion Theorem within this structure in order to show that F is definable over $\{\langle \mathrm{H}(\kappa), \in, \lhd_\kappa \rangle\}$ by a formula without parameters. In combination, these observations yield the statement of the claim. □

Now, let $\mathbb{P}$ be a partial order of cardinality less than δ such that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and let G be $\mathbb{P}$-generic over V. By Definition 7.1.(ii), $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameters κ and ν. By the above claim and Σ_1-absoluteness, the set $\{\langle S_\beta \mid \beta < \kappa \rangle\}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}}, \in \rangle$

by a Σ_1-formula with parameters κ and ν. Together, these statements imply that $\{\langle S_\beta \mid \beta < \kappa\rangle\}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν.

This completes the proof of the theorem. □

Corollary 7.8. *Let η be an infinite regular cardinal and let $\langle \delta, \theta, \nu, \lhd\rangle$ be suitable for $\kappa = \eta^+$. Then there is a sequence $\langle S_\alpha \mid \alpha \leq \kappa\rangle$ of pairwise disjoint stationary subsets of S^κ_η with the property that the set $\{\langle S_\alpha \mid \alpha < \kappa\rangle\}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν whenever $\mathbb{P}$ is a partial order of cardinality less than δ such that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and G is $\mathbb{P}$-generic over* V.

Proof. Let $\langle \bar{S}_\alpha \mid \alpha \leq \kappa\rangle$ be the sequence of pairwise disjoint fat stationary subsets of κ produced by Theorem 7.7. Given $\alpha \leq \kappa$, set $S_\alpha = \bar{S}_\alpha \cap S^\kappa_\eta$. Then S_α is a stationary subset of κ for each $\alpha \leq \kappa$.

Let $\mathbb{P}$ be a partial order of cardinality less than δ such that forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and let G be $\mathbb{P}$-generic over V. By Lemma 7.6.(vi) and Σ_1-reflection, the set $\{S^\kappa_\eta\}$ is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν. Since Theorem 7.7 shows that the set $\{\langle \bar{S}_\alpha \mid \alpha < \kappa\rangle\}$ is definable in the same way, this yields the statement of the corollary. □

Corollary 7.9. *Assume that κ is either the successor of a regular cardinal or an inaccessible cardinal. Let $\langle \delta, \theta, \nu, \lhd\rangle$ be suitable for κ and let $\mathbb{P}$ be a partial order of cardinality less than δ with the following properties.*

- *(a) Forcing with $\mathbb{P}$ preserves cofinalities less than or equal to κ^+ and fat stationary subsets of κ.*
- *(b) If G is $\mathbb{P}$-generic over* V*, then 2^κ is regular, $\kappa = \kappa^{<\kappa}$ and $\eta^{<\eta} < \kappa$ for all $\eta < \kappa$ in* V$[G]$*.*

Then there is a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order such that the following statements hold whenever $G * H$ is $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V*.*

- *(i) The partial order $\dot{\mathbb{Q}}^G$ is $<\kappa$-distributive in* V$[G]$*.*
- *(ii) Forcing with $\dot{\mathbb{Q}}^G$ over* V$[G]$ *preserves all cofinalities less than or equal to $(2^\kappa)^{\mathrm{V}[G]}$ and the value of 2^κ.*
- *(iii) There is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}$ that is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν.*

Proof. If κ is the successor of a regular cardinal η, let $\vec{S} = \langle S_\alpha \mid \alpha \leq \kappa\rangle$ denote the sequence of pairwise disjoint stationary subsets of S^κ_η produced by Corollary 7.8. If κ is an inaccessible cardinal, then we let $\vec{S}$ denote the sequence of fat stationary subsets of κ provided by Theorem 7.7. In either case, by (a), $\vec{S}$ is a sequence of either fat stationary subsets of κ or stationary subsets of S^κ_η respectively in every $\mathbb{P}$-generic extension of the ground model. Together with (b), this shows that κ and $\vec{S}$ satisfy the requirements of Theorem 1.4 in every $\mathbb{P}$-generic extension of the ground model. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for the partial order given by Theorem 1.4. Since $\mathbb{P}$ has cardinality less than δ and δ is a strong limit cardinal, the construction of the forcing in the proof of Theorem 1.4 shows that we can find such a name with the property that $\mathbb{P} * \dot{\mathbb{Q}}$ also has cardinality less than δ.

Let $G * H$ be $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V. By Theorem 1.4, $\dot{\mathbb{Q}}^G$ is ${<}\kappa$-distributive in $\mathrm{V}[G]$, forcing with $\dot{\mathbb{Q}}^G$ over $\mathrm{V}[G]$ preserves all cofinalities less than or equal to $(2^\kappa)^{\mathrm{V}[G]}$ and the value of 2^κ and there is a well-ordering $\blacktriangleleft$ of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}, \in\rangle$ by a Σ_1-formula with parameter $\langle S_\alpha \mid \alpha < \kappa\rangle$. Since Theorem 7.7 shows that the set $\{\langle S_\alpha \mid \alpha < \kappa\rangle\}$ is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν, we can conclude that the well-order $\blacktriangleleft$ is also definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}, \in\rangle$ by a Σ_1-formula with parameters κ and ν. □

Proof of Theorem 1.5. Assume that $\mathrm{V} = \mathrm{L}$ holds and κ is either the successor of a regular cardinal or an inaccessible cardinal. Let $\mathbb{P}$ be a partial order with the properties (a) and (b) listed in Corollary 7.9. Pick a strong limit cardinal δ with $|\mathbb{P}| < \delta$. By Lemma 7.3, there is a well-order $\lhd$ of $\mathrm{H}(\kappa^+)$ such that the tuple $\langle \delta, \kappa^+, 0, \lhd\rangle$ is suitable for κ. Let $\dot{\mathbb{Q}}$ be the $\mathbb{P}$-name for a partial order given by Corollary 7.9. Then Corollary 7.9 implies that the statements (i)-(iii) listed in the theorem hold. □

Proof of Theorem 1.7. Assume that U is a normal measure on δ, $\mathrm{V} = \mathrm{L}[U]$ holds and $\kappa < \delta$ is either the successor of a regular cardinal or an inaccessible cardinal. Let $\mathbb{P} \in \mathrm{V}_\delta$ be a partial order with the properties (a) and (b) listed in Corollary 7.9. By Lemma 7.5, there is a well-order $\lhd$ of $\mathrm{H}(\kappa^+)$ such that the tuple $\langle \delta, \kappa^+, \kappa, \lhd\rangle$ is suitable for κ. Let $\dot{\mathbb{Q}}$ be the $\mathbb{P}$-name for a partial order given by Corollary 7.9 and let $G * H$ be $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V. Then Corollary 7.9 implies that the partial order $\dot{\mathbb{Q}}^G$ is ${<}\kappa$-distributive in $\mathrm{V}[G]$, forcing with $\dot{\mathbb{Q}}^G$ over $\mathrm{V}[G]$ preserves all cofinalities less than or equal to $(2^\kappa)^{\mathrm{V}[G]}$ and the value of 2^κ and there is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}$ that is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G,H]}, \in\rangle$ by a Σ_1-formula with parameter κ. □

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