Characterizing Large Cardinals through Forcing

Peter Holy

University of Bonn

presenting joint work with Philipp Lücke and Ana Njegomir

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Introduction

Many combinatorial properties on accessible cardinals can be established by collapsing a certain large cardinal to become one of the $\aleph_n$'s, $n < \omega$. For example, if we turn a Mahlo cardinal into a successor cardinal κ^+ by a standard Lévy collapse, we obtain a failure of $\Box_\kappa$. We are interested in the question in which situations it is possible to also recognize previous large cardinals in collapse extensions through the validity of certain combinatorial principles at the collapsed cardinal.

The Lévy collapse

The Lévy collapse forcing P cannot be used to characterize any kind of large cardinal. The problem is that $P \simeq P \times P \simeq P * \check{P}$, so if any formula provably were to hold after a Lévy collapse if and only if that forcing were applied to a certain kind of large cardinal, this would immediately be seen to fail in an intermediate extension for the same Lévy collapse forcing.

Question

Is there a way around this?

The easiest way to obtain positive characterization results seems to be to use different collapse forcing notions!

Positive Results

We use Neeman's pure side condition forcing to characterize the following large cardinals through well-known combinatorial principles, and through what we call *internal large cardinal properties*:

- Inaccessible and Mahlo Cardinals
- Π_n^m-indescribable cardinals
- Subtle and λ-ineffable cardinals

We then use generic large cardinal properties to characterize the following:

- Measurable and γ-supercompact cardinals
- Almost huge and super almost huge cardinals

Neeman's pure side condition forcing

Let θ be an infinite cardinal. We define the forcing P_θ as follows:

- $\mathcal{S}_\theta$ is the set of all countable elementary submodels of $H(\theta)$ in $H(\theta)$.
- $\mathcal{T}_\theta$ is the set of all transitive and countably closed elementary submodels of $H(\theta)$ in $H(\theta)$.
- Conditions in P_θ are finite, $\in$-increasing sequences $\langle M_i \mid i < n\rangle$ of elements of $\mathcal{S}_\theta \cup \mathcal{T}_\theta$, closed under intersections.
- We order P_θ through reverse inclusion on the range of its conditions.

The basis case

We call an infinite cardinal θ *countably inaccessible* if it is regular and $\delta^\omega < \theta$ for every $\delta < \theta$. For such θ, P_θ preserves both ω_1 and θ, collapses all cardinals inbetween, and satisfies the σ-approximation property (this is essentially all due to Neeman). The next lemma then follows easily:

Lemma

The following are equivalent:

1. θ is countably inaccessible.
2. $P_\theta \Vdash \check{\theta} = \omega_2$.

This is already a characterization of a (not quite large) cardinal property through Neeman's forcing!

Inaccessible Cardinals

A tree of height ω_1 and cardinality $\aleph_1$ is a *weak Kurepa tree* if it has at least $\aleph_2$-many branches. If CH holds, the full binary tree of height ω_1 is a weak Kurepa tree. However, CH fails in our situation. Adapting fairly standard arguments yields the following:

Lemma

The following statements are equivalent for every countably inaccessible cardinal θ:

1. θ is an inaccessible cardinal
2. $P_\theta \Vdash$ there are no weak Kurepa trees.

Combining this with the result from the previous slide yields a characterization of inaccessible cardinals.

Generic Large Cardinals

Making strong use of ideas from Viale and Weiß, one obtains the following:

Theorem

The following are equivalent for every inaccessible cardinal θ and every cardinal $\gamma \geq \theta$:

1. θ is a γ-supercompact cardinal.
2. $P_\theta \Vdash$ there is a partial order with the σ-approximation property that witnesses ω_2 to be generically γ-supercompact.

Theorem

Similarly, Neeman's forcing can be used to characterize almost huge and super almost huge cardinals through their generic counterparts.

Small Embedding Characterizations of Large Cardinals

For substantially smaller large cardinals, we came up with the concept of *internal large cardinals* in order to provide somewhat analogous characterizations. This concept is based on *small embedding characterizations for large cardinals*: A small embedding for θ is an elementary embedding $j\colon M \to H(\nu)$ for $M \in H(\nu)$ transitive, with $j(\mathrm{crit}\, j) = \theta$. A classical small embedding characterization of a large cardinal is due to Magidor:

Theorem (Magidor)

θ is supercompact if and only if for every $\nu > \theta$ there is $\alpha < \theta$ and $j\colon H(\alpha) \to H(\nu)$ with $j(\mathrm{crit}\, j) = \theta$.

Further small embedding characterizations

However, many further large cardinal notions admit a small embedding characterization. For example, we have the following folklore result, rephrased in our terminology:

Lemma

The following are equivalent for every regular uncountable cardinal θ:

1. θ is a Mahlo cardinal.
2. For every $\nu > \theta$, there is a small embedding $j\colon M \to H(\nu)$ for θ such that $\operatorname{crit} j$ is regular and $H(\operatorname{crit} j) \subseteq M$.

Remark 1: $\operatorname{crit} j$ regular and $H(\operatorname{crit} j) \subseteq M$ is a *correctness property* between M and V.
Remark 2: We can additionally require any given $x \in H(\nu)$ to be in the range of j in (2) above.

Now there is a fairly uniform way to turn small embedding characterizations of large cardinals into a definition for an internal large cardinal concept. We require the existence of an inner model $N \subseteq H(\nu)$, we require that $M \in N$, we relativize the required correctness property between M and V to N, and – for the case of Neeman's forcing – we require that the pair $(N, H(\nu))$ satisfies the σ-approximation property.

DRAW A PICTURE!

Internally Mahlo cardinals...

Definition

θ is *internally Mahlo* if for all sufficiently large regular cardinals ν and all $x \in H(\nu)$, there is a small embedding $j\colon M \to H(\nu)$ for θ with $x \in \operatorname{range} j$, and a transitive model N of ZFC^-, such that the following statements hold:

1. $N \subseteq H(\nu)$, and the pair $(N, H(\nu))$ satisfies the σ-approximation property.
2. $M \in N$, and $\mathcal{P}_{\omega_1}(\operatorname{crit} j)^N \subseteq M$.
3. $\operatorname{crit} j$ is a regular cardinal in N.

Characterizing Mahlo cardinals

Lemma (Easy)

If θ is internally Mahlo, then θ is regular and uncountable, and there are no special θ-Aronszajn trees.

Theorem

The following are equivalent for every inaccessible cardinal θ:

1. θ is a Mahlo cardinal.
2. $P_\theta \Vdash \omega_2$ is internally Mahlo.
3. $P_\theta \Vdash$ there are no special ω_2-Aronszajn trees.

Proof:

(1) $\rightarrow$ (2): Lift the embedding.
(2) $\rightarrow$ (3): The above lemma.
(3) $\rightarrow$ (1): Follows easily from results by Todorčević, making use of his generalized notion of special Aronszajn trees at inaccessible cardinals.

Internally indescribable cardinals

Based on a small embedding characterization of Π^m_n-indescribable cardinals, we obtain internally Π^m_n-indescribable cardinals:

Definition

A cardinal θ is *internally Π^m_n-indescribable* if for all sufficiently large regular cardinals ν and all $x \in H(\nu)$, there is a small embedding $j\colon M \to H(\nu)$ for θ, and a transitive $N \models \mathrm{ZFC}^-$ such that $x \in \operatorname{range} j$, and the following hold:

1. $N \subseteq H(\nu)$, and the pair $(N, H(\nu))$ satisfies the σ-approximation property.
2. $M \in N$, and $P_{\omega_1}(\operatorname{crit} j)^N \subseteq M$.
3. $\operatorname{crit} j$ is regular in N and

$$(H(\operatorname{crit} j) \models \Phi(A))^M \ \to\ (H(\operatorname{crit} j) \models \Phi(A))^N$$

for every Π^m_n-formula Φ with parameter $A \in \mathcal{P}(H(\operatorname{crit} j))^M$.

Π^1_1-indescribables

Lemma (Easy)

If θ is internally Π^1_1-indescribable, then θ is regular and uncountable and the tree property holds at θ.

Theorem

The following are equivalent for every inaccessible cardinal θ:

1. *θ is a Π^1_1-indescribable cardinal.*
2. *$P_\theta \Vdash \omega_2$ is internally Π^1_1-indescribable.*
3. *$P_\theta \Vdash \omega_2$ has the tree property.*

The equivalence between (1) and (3) is essentially due to Neeman.

Π^m_n-indescribables

Lacking a combinatorial analogue of the tree property with respect to higher levels of indescribability, the following result perhaps suggests internal indescribability to serve as such analogue.

Theorem

The following are equivalent for every inaccessible cardinal θ.

1. *θ is a Π^m_n-indescribable cardinal.*
2. *$P_\theta \Vdash \omega_2$ is internally Π^m_n-indescribable.*

Moreover, if some cardinal θ is internally Π^m_n-indescribable and inaccessible, then θ is in fact already Π^m_n-indescribable.

Other large cardinals

One can similarly define notions of internal large cardinals for:

- Subtle cardinals.
- λ-ineffable cardinals.
- Supercompact cardinals.

These turn out to be closely related to notions introduced and studied by Viale and Weiß. They allow for characterizations of the respective large cardinals similar to above.