Large Cardinals and lightface definable Wellorders without GCH

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presenting joint work with Sy Friedman and Philipp Lücke

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Locally definable wellorders

$X \subseteq H_\kappa$ is *lightface / boldface definable over* H_κ if X is definable over $\langle H_\kappa, \in\rangle$ by a formula without parameters / by a formula with parameter $p \in H_\kappa$.
We say that there is a *lightface / boldface definable wellorder of* H_κ if there is a wellorder of H_κ which is lightface / boldface definable over H_κ.

Theorem (Gödel, 1935)

There is a Δ_1-definable relation $<_L$ providing a wellorder of L and for every limit ordinal α, its restriction to L_α is a lightface (Δ_1-)definable wellorder of L_α. In particular if $V = L$, then for every κ there is a lightface definable wellorder of H_κ.

Motivating Questions

This gives rise to the question to what extent definable wellorders of H_κ can coexist with principles contradicting $V = L$. We will focus on the existence of (very) large cardinals and failures of the GCH (note that H_κ has size $2^{<\kappa}$, therefore failures of the GCH necessitate long wellorders of H_κ). A classic result by Leo Harrington is the following:

Theorem (Harrington, 1977)

It is consistent to have $2^{\aleph_0}$ as large as desired while there is a lightface definable wellorder of H_{ω_1}. This is shown by forcing over a model with $\aleph_1 = {\aleph_1}^L$.

Assuming GCH

Assuming GCH, the following has been shown:

Theorem (Aspero - Friedman, 2009)

Assume GCH. Then there is a cofinality-preserving forcing which introduces a lightface definable wellorder of H_{κ^+} for every regular uncountable κ, preserving the GCH. Moreover all inaccessibles, all instances of supercompactness and many other large cardinal properties are preserved.

H_{ω_1} is not included in the above result. It is indeed known that the existence of large cardinals prohibits the existence of definable wellorders of H_{ω_1}:

Theorem (Martin - Steel, 1985)

If there are infinitely many Woodin cardinals, then Projective Determinacy holds. Latter implies that there is no definable wellorder of H_{ω_1}.

Singular Cardinals

When κ is singular (and of cofinality ω), the existence of definable wellorders of H_{κ^+} is also connected to the existence of large cardinals:

Theorem (Aspero - Friedman 2009)

If κ is an I0 cardinal, i.e. if there is j with $\lambda = j^\omega(\kappa)$, $\kappa = \mathrm{crit}(j)$ and $j\colon L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, then there is no definable wellorder of H_{λ^+}.

If κ is a limit cardinal, definable wellorders of H_κ are easily induced by lightface definable wellorders of H_λ for $\lambda < \kappa$:

Fact

If κ is a limit cardinal and for unboundedly many $\lambda < \kappa$, H_λ has a lightface definable wellorder, then H_κ has a lightface definable wellorder.

For the rest of this talk, I want to consider definable wellorders of H_{κ^+} for κ regular and uncountable in the non-GCH case.

Forcing boldface definable wellorders

Theorem (Lücke, 2012)

If κ is regular and uncountable with $\kappa^{<\kappa} = \kappa$, then there is a partial order (the generic tree coding partial order) P which is κ^+-cc and $<\kappa$-closed and forces that there exists a boldface definable wellorder of H_{κ^+}.

This is done by choosing a wellorder $\leq$ of H_{κ^+} and then generically adding a subtree of ${}^{<\kappa}2$ the branches of which code $\leq$. This forcing P has the property that P itself and its generic G are definable over $H_{\kappa^+}^{V[G]}$. As P is κ^+-cc, every element of $H_{\kappa^+}^{V[G]}$ has a name in H_{κ^+}. Hence $\leq$ induces a (definable) wellordering of $H_{\kappa^+}^{V[G]}$:

$$x \leq^* y \iff \dot{x} \leq \dot{y}$$

where $\dot{x}$ is the $\leq$-least P-name such that $\dot{x}^G = x$
and $\dot{y}$ is the $\leq$-least P-name such that $\dot{y}^G = y$.

Prohibiting boldface definable wellorders

An observation (which follows from a theorem of Hugh Woodin) that we will make use of later on is that the existence of boldface definable wellorders can in fact be *switched on and off* by mild forcing:

Lemma (Lücke, 2012)

If κ is regular and uncountable with $\kappa^{<\kappa} = \kappa$, then after adding κ^+-many Cohen subsets of κ, there is no boldface definable wellorder of H_{κ^+}. Moreover $\mathrm{Add}(\kappa, \kappa^+)$ is κ^+-cc and $<\kappa$-closed.

A global boldface result

For boldface definable wellorders, Sy Friedman and Philipp Lücke obtained the following global result, assuming SCH (but not GCH):

Theorem (Friedman - Lücke, 2012)

Assume SCH. There is a class forcing P with the following properties:

- *P preserves all inaccessibles and all supercompacts.*
- *Whenever κ is inaccessible, P introduces a boldface definable wellorder of H_{κ^+}.*
- *P is cofinality-preserving and preserves the continuum function.*

Most of the above also holds without assuming SCH, but it is unknown in that case whether or not the above iteration collapses *counterexamples to the SCH*.

Boldface $\longrightarrow$ Lightface

To what extent can 'boldface' be replaced by 'lightface' in the above? Our strategy will be to first obtain a boldface definable wellorder of some H_{κ^+} using generic tree coding and then perform a further forcing to *turn it into* a lightface definable wellorder of H_{κ^+}. We need the following:

Lemma

Assume $\leq$ is a boldface definable wellorder of H_{κ^+} with parameter p for some regular κ. Let P be κ^+-cc and let G be P-generic. Assume P, G, $H^V_{\kappa^+}$ and p are lightface definable over $H^{V[G]}_{\kappa^+}$. Then in $V[G]$, there is a lightface definable wellorder $\leq^*$ of H_{κ^+} s.t.

$$x \leq^* y \iff \dot{x} \leq \dot{y}$$

where $\dot{x}$ is the $\leq$-least P-name such that $\dot{x}^G = x$ and $\dot{y}$ is the $\leq$-least P-name such that $\dot{y}^G = y$.

Local

Let us first recall the boldface theorem:

Theorem (Lücke, 2012)

If κ is regular and uncountable with $\kappa^{<\kappa} = \kappa$, then there is a partial order which is κ^+-cc and $<\kappa$-closed and forces that there exists a boldface definable wellorder of H_{κ^+}.

We will obtain the following *lightface version* of this theorem:

Lemma

Assume SCH. If κ is inaccessible, then there is a partial order which is κ^+-cc, preserves cofinalities and the continuum function and forces that there exists a lightface definable wellorder of H_{κ^+}.

Proof: Use generic tree coding to introduce a boldface definable wellorder of H_{κ^+} with parameter $p \in H_{\kappa^+}$. We want to turn this into a lightface definable wellorder of H_{κ^+} - thus we have to find a forcing P which makes p, P, H_{κ^+} and the P-generic lightface definable in the H_{κ^+} of the P-generic extension.

Proof continued

Let Γ denote the class of successors of singular fix points of the $\beth$-function. $\Gamma \cap \kappa$ has cardinality κ and by the SCH, $\lambda^{<\lambda} = \lambda$ for every $\lambda \in \Gamma$. Let $\langle b_i \mid i < \kappa \rangle$ be the increasing enumeration of $\Gamma \cap \kappa$. We may assume that $p \subseteq \kappa$. Now perform a reverse Easton iteration of length κ which at stage $i < \kappa$ makes sure that $H_{b_i^+}$ has a boldface definable wellorder iff $i \in p$. By the closure properties of tails of the iteration, this will still be true in the final generic extension and hence we made p lightface definable in H_{κ^+}. For H_{κ^+}, this follows from absoluteness properties of the generic tree coding and P is simply lightface definable in the ground model H_{κ^+}. To make the P-generic definable, we have to use a slightly more complicated forcing Q (instead of P) where odd stages of the iteration are used to code the generic. This can be done using a simple book-keeping argument as P (or also Q) has size κ and conditions in P (or also Q) have support bounded in κ. $\square$

Global & supercompacts

Let us first recall the boldface theorem:

Theorem (Friedman - Lücke, 2012)

Assume SCH. There is a class forcing P with the following properties:

- *P preserves all inaccessibles and all supercompacts.*
- *Whenever κ is inaccessible, P introduces a boldface definable wellorder of H_{κ^+}.*
- *P is cofinality-preserving and preserves the continuum function.*

We obtain the following *lightface version* of this theorem:

Theorem (F-H-L)

Assume SCH. There is a class forcing P with the following properties:

- *P preserves all inaccessibles and all supercompacts.*
- *Whenever κ is inaccessible, P introduces a lightface definable wellorder of H_{κ^+}.*
- *P is cofinality-preserving and preserves the continuum function.*

Easy?

Before giving a sketch of the proof of this result, we want to indicate why this global result cannot be obtained by just iterating (in some way) the local result from the previous slide.

For every inaccessible cardinal κ, we could try to code some $x_\kappa \subseteq \kappa$ by switching the existence of boldface definable wellorders of H_{λ^+} on and off for $\lambda \in A_\kappa \subseteq \kappa$, with A_κ of size κ.

Assume θ is inaccessible with stationary many inaccessibles below (i.e. θ is Mahlo). Using Fodor's Theorem, it follows that

$$\bigcap_{\kappa<\theta} A_\kappa \neq \emptyset.$$

Therefore we cannot expect such a coding iteration to work as the areas into which we want to code potentially different information overlap.

A different strategy

We have to use a more uniform approach to this problem. For simplicity, we will again consider just a single inaccessible cardinal κ and present a different way of introducing a lightface definable wellorder of H_{κ^+} which can - in some sense - be iterated. Let Γ denote the successor cardinals of the singular fix points of the $\beth$-function. Assume SCH holds.

Our strategy:

- Add a Cohen subset c_κ of κ and code it into the *boldface definable wellorder existence pattern* on $\Gamma \cap \kappa$ by a reverse Easton iteration.
- Use generic tree coding to introduce a boldface definable wellorder of H_{κ^+} with parameter $x_\kappa \subseteq \kappa$.
- Force with a forcing CF_κ to ensure that c_κ codes x_κ and the generic for CF_κ.

Let P_κ denote this three step iteration.

Canonical Function Coding (Aspero - Friedman)

For every inaccessible κ, uniformely choose a recursive sequence $\langle f_\gamma \mid \gamma \in [\kappa, \kappa+\kappa)\rangle$ such that each f_γ is a bijection from κ to γ. Fix some inaccessible κ and $\gamma = \kappa + \delta$ with $\delta < \kappa$. Then $\{i < \kappa \mid c_\kappa(\text{ot } f_\gamma[i]) = 0\}$ is stationary and co-stationary. We want to shoot a club (using as conditions closed, bounded subsets of κ, ordered by end-extension) through either the above set or its complement to ensure that in the generic extension,

$$\{i < \kappa \mid c_\kappa(\text{ot } f_\gamma[i]) = x_\kappa(\delta)\}$$

contains a club. CF_κ will be an iteration with $<\kappa$-support of such club shooting forcings to ensure that x_κ is coded by c_κ. One has to verify that P_κ is κ^+-cc, preserves all cofinalities and the continuum function and that components in CF_κ can be extended to have arbitrary large maximum below κ.

Canonical Function Coding 2

This will ensure that x_κ is lightface definable in the H_{κ^+} of the P-generic extension. Let W denote the generic extension after forcing with the first two iterands of P, which are adding the wellorder-existence coded Cohen subset of κ and the generic tree coding forcing to introduce a boldface definable wellorder of H_{κ^+}. To obtain a lightface definable wellorder of H_{κ^+} in the CF_κ-generic extension M, we also have to ensure that CF_κ, the generic for CF_κ and $H^W_{\kappa^+}$ are lightface definable in $H^M_{\kappa^+}$. For $H^W_{\kappa^+}$, this follows by absoluteness properties of the generic tree coding and CF_κ is simply lightface definable in $H^W_{\kappa^+}$. For the CF_κ-generic, this is done by a bookkeeping argument, using that CF_κ has - in a sense - a dense subset of size κ and conditions have support bounded in κ - therefore the generic can also be coded within the above iteration.

The global iteration

Now we want to iterate - in some sense - P_κ for all inaccessible κ with a reverse Easton iteration. Latter will allow for the preservation of all supercompact cardinals. To define the global iteration, we will describe what to do at each stage $\alpha \in \mathrm{Ord}$:

The global iteration:

- $\alpha \in \Gamma$
  - Stage α: Let the generic choose between 0 or 1.
  - Stage $\alpha + 1$: Prohibit (if 0 was chosen) or enforce (otherwise) a boldface definable wellorder of H_{α^+}.
- α is inaccessible
  - Stage α: Enforce a boldface definable wellorder of H_{α^+}.
  - Stages $\alpha + 1$ to $\alpha + \alpha$: Perform a slight variant of the Canonical Function Coding iteration CF_α with $<\alpha$-support.
- otherwise perform the trivial forcing at stage α.

Other Large Cardinals

This answers one of the motivating questions of this work - whether one can get a measurable κ with $2^\kappa > \kappa^{++}$ and a lightface definable wellorder of H_{κ^+}. For $2^\kappa = \kappa^{++}$, this was done by Sy Friedman and Radek Honzik by forcing over the canonical inner model for a κ^{++}-strong cardinal.

Theorem (F-H-L)

Assuming the consistency of a supercompact cardinal, it is consistent to have a measurable cardinal κ with $2^\kappa > \kappa^{++}$ and a lightface definable wellorder of H_{κ^+}.

Also, whenever κ is ω-superstrong, there is a condition in our three-step iteration forcing that the ω-superstrength of κ is preserved.

Theorem (F-H-L)

Given an ω-superstrong cardinal, it is consistent to have an ω-superstrong κ with 'arbitrary' 2^κ and a lightface definable wellorder of H_{κ^+}.

A sideresult: V=HOD

As can easily be seen from the definition, our iteration adds lightface definable wellorders of H_{κ^+} for unboundedly many cardinals κ, which moreover are uniformely definable over H_{κ^+}. This clearly implies V=HOD. Therefore we obtain as a side result that V=HOD can be forced over any model of SCH by cofinality- and continuum function preserving forcing which preserves all inaccessibles and all supercompact cardinals. If we leave out the iterations CF_α from our iteration (which are not necessary to obtain V=HOD) and leave sufficiently large gaps between places where we perform non-trivial forcing (this still gives V=HOD), we can also preserve many other large cardinals. This has been done in great detail by Andrew Brooke-Taylor under the assumption of and preserving GCH (switching on and off $\diamondsuit^*_\kappa$ for various κ). Our methods (switching on and off the existence of boldface definable wellorders of H_{κ^+} for various κ) generalize this to non-GCH situations, just assuming SCH.

Open Questions

Question

Can one get the consistency of a measurable κ with $2^\kappa > \kappa^{++}$ and a lightface definable wellorder of H_{κ^+} from a large cardinal assumption weaker than (some level of) supercompactness?

Question

Is there a 'nice' (in particular cofinality-preserving) forcing to introduce a lightface definable wellorder of H_{κ^+} when $\kappa^{<\kappa} = \kappa$ but κ is not inaccessible (assuming SCH)?

Question

Is it possible to get rid of the SCH assumption in some of the theorems?

Thank you.