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An introduction to local models of Shimura varieties

In this talk, a Shimura var. is the solution of a moduli problem of abelian varieties (w. polariz., endo's, level structures) over a number field E. Call it $\mathcal{M}_E$

Assume level str. at p is parahoric.

Problem: Fix a prime ideal $\mathfrak{p}$ of E of res. char p. Find a model $\mathcal{M}$ of $\mathcal{M}_E$ over $\operatorname{Spec} \mathcal{O}_{E_\mathfrak{p}}$, i.e.

$$\mathcal{M} \times_{\operatorname{Spec} \mathcal{O}_{E_\mathfrak{p}}} \operatorname{Spec} E_\mathfrak{p} = \mathcal{M}_E \times_{\operatorname{Spec} E} \operatorname{Spec} E_\mathfrak{p}$$

Want a model which is flat, has rel. simple singularities, has description of $\mathcal{M}(\bar{\kappa}_\mathfrak{p})$. Also is projective, if $\mathcal{M}_E$ is.

Naive idea (Rapo-Zink): Extend the moduli functor from $(Sch / E_\mathfrak{p})$ to $(Sch / \mathcal{O}_{E_\mathfrak{p}})$, "in obvious way". This works often - and even when it doesn't work, want the flat closure inside naive model.

To investigate the singularities of $\mathcal{M}$ use local model + their relation to certain matrix singularities.

Motivation: a) Semi-simple zeta function
b) Arithm. theory of modular forms.

1. The Ur-example: moduli of elliptic curves.

Fix $m \geq 3$, $(m, p) = 1$ (auxiliary). Put $F = \mathbb{Q}$. Define functor $\mathcal{M}_{\mathbb{Q}}$ on $(Sch/\mathbb{Q})$ by

$$\mathcal{M}_{\mathbb{Q}}(S) = \{ (\underset{\text{elliptic curve}/S}{A}, \alpha), \quad \alpha : A_m \simeq (\mathbb{Z}/m)^2 \} / \simeq .$$

This is repr. by quasi-proj. scheme smooth of rel. dim 1 over Spec $\mathbb{Q}$.

Same formulation gives functor $\mathcal{M}$ on $(Sch/\mathbb{Z}_p)$, again repr. by quasi-proj. schem over Spec $\mathbb{Z}_p$.

Thm: $\mathcal{M}$ is smooth of rel. dim 1 over Spec $\mathbb{Z}_p$.

Fancy proof: Have diagram

$$\begin{tikzcd} & \mathcal{M}^{\natural} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M} & & \mathbb{P}^1 \end{tikzcd}$$

Here in $\mathcal{M}^{\natural}$ add to (A, α) a basis of $H_1^{DR}(A)$,

$$\eta : \mathcal{O}_S^2 \xrightarrow{\sim} H_1^{DR}(A).$$

Hence π is phs under GL_2. Then φ maps (A, α, η) to

$$[\mathcal{F} = \eta^{-1}(\omega_{A/S}) \subset \mathcal{O}_S^2] \in \mathbb{P}^1(S).$$

By Serre-Tate + Groth-Messing, φ satisf. the i-finit. criterion for smoothness.

Here $\mathbb{P}^1$ was the <u>local model for this moduli problem</u> (<u>ag. linear</u>)

<u>Next let us consider</u> $\Gamma_0(p)$ - <u>moduli problem</u>. Again $E = \mathbb{Q}$, $n \geq 3$.

$$\mathcal{M}(S) = \{ (A_0 \xrightarrow{\psi} A_1, \alpha), \ \psi \text{ isogeny of degree } p \} / \simeq .$$

Can again extend this to quasi-proj. scheme over Spec $\mathbb{Z}_p$.

<u>Thm</u>: $\mathcal{M}$ is flat of relative dimension 1 over Spec $\mathbb{Z}_p$ and has semistable reduction.

<u>Proof</u>: As before get diagram with smooth morphisms (π plus under smooth gp scheme).

$$\begin{tikzcd} & \mathcal{M}^{4} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M} & & \mathrm{M} \end{tikzcd}$$

Here M is the <u>local model</u>, repr. following functor over Spec $\mathbb{Z}_p$.

Let $V = \mathbb{Q}_p^2$ with std basis e_1, e_2. Let

$$\Lambda_0 = \mathrm{span}_{\mathbb{Z}_p}\{e_1, e_2\} \quad , \quad \Lambda_1 = \mathrm{span}_{\mathbb{Z}_p}\{p^{-1}e_1, e_2\}$$

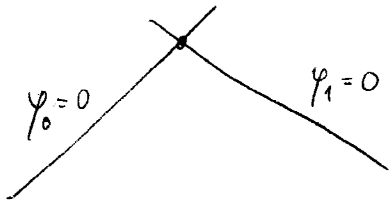
$$\Lambda_0 \longrightarrow \Lambda_1 \xrightarrow{p} \Lambda_0 .$$

Then $M(S) =$ iso-classes of commut. diagrams

$$\begin{array}{ccccc}
\Lambda_{0,S} & \longrightarrow & \Lambda_{1,S} & \longrightarrow & \Lambda_{0,S} \\
\cup & & \cup & & \cup \\
\mathcal{F}_0 & \xrightarrow{\varphi_0} & \mathcal{F}_1 & \xrightarrow{\varphi_1} & \mathcal{F}_0
\end{array}$$

where $\mathcal{F}_0, \mathcal{F}_1$ are loc. direct summands free of rank 1.

<u>Picture of</u> $M \otimes \mathbb{F}_p$:



Choose coord. locally around worst point

$$(\mathcal{F}_0^0, \mathcal{F}_1^0) = (p\Lambda_1, \Lambda_0) \in M(\mathbb{F}_p).$$

Get equ.: $X_0 \cdot X_1 = p.$

In the remainder I want to discuss an assortment of examples.

For efficiency, I will not formulate precisely the moduli problem $\mathcal{M}$, but concentrate on local models.

<u>Naive idea works in „unramified situation", but not in „ramified situation"</u>

2. The unramified unitary group

$\mathcal{M}_E$ $\left\{\begin{array}{l}\text{Let } n = r+s. \text{ Let } L = \text{imaginary-quadr. field} \overset{c}{\subset} \mathbb{C}, \text{ consider moduli} \\ \text{space of abelian var. } A \text{ of dim. } n, \text{ with action of } O_L, \text{ and } O_L\text{-linear} \\ \text{principal polarization s.t. } \operatorname{tr}(\iota(b), \operatorname{Lie} A) = r\cdot b + s\cdot \bar{b}, \ \forall\, b\in O_L. \\ + \text{ level structure away from } p + \text{ parahoric level structure at } p. \\ \text{In this case } E = \begin{cases} L & r\neq s \\ \mathbb{Q} & r = s \end{cases}. \end{array}\right.$

If p is unramified in L, the naive idea leads to following local model M. Let $V = \mathbb{Q}_p^n$ with basis $e_1, e_2, \dots, e_n$.

For $i = 0, \dots, n-1$, let $\Lambda_i = \operatorname{span}_{\mathbb{Z}_p}\{p^{-1}e_1, \dots, p^{-1}e_i, e_{i+1}, \dots, e_n\}$.

Get

$$\Lambda_0 \longrightarrow \Lambda_1 \longrightarrow \dots \longrightarrow \Lambda_{n-1} \xrightarrow{\ p\ } \Lambda_0$$

Choose $I = \{i_0 < i_1 < \dots < i_{m-1}\} \subset \{0, \dots, n-1\}$. Then $M = M_I$ is described as $M(S) =$ iso-classes of commut. diagr.

$$\begin{array}{ccccccccc} \Lambda_{i_0,S} & \longrightarrow & \Lambda_{i_1,S} & \longrightarrow & \dots & \longrightarrow & \Lambda_{i_{m-1},S} & \xrightarrow{\ p\ } & \Lambda_{i_0,S} \\ \cup & & \cup & & & & \cup & & \cup \\ \mathcal{F}_0 & \xrightarrow{\ \varphi_0\ } & \mathcal{F}_1 & \xrightarrow{\ \varphi_1\ } & \dots & \longrightarrow & \mathcal{F}_{m-1} & \longrightarrow & \mathcal{F}_0 \end{array}$$

Here $\mathcal{F}_i$ are loc. direct summands free of rank r.

For $|I| = 1$, M_I is the Grassmannian $Gr_{r,n-r}$ over $\mathbb{Z}_p$.
Contrast to classical moduli pb's, like variety of complexes etc.: space fixed, maps vary
In general have diagram $\Updownarrow$ maps fixed, spaces vary.

$$\begin{tikzcd}
 & M^{\square} \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\
M & & N
\end{tikzcd}$$

where for $M^{\square}$ we also fix a basis for $\mathcal{F}_i$, $\forall i \in I$.

Hence π is pls. for $\prod_{i\in I} GL_r$. And

$$N_{r,m} = \{ (A_0, A_1, \ldots, A_{m-1}) \in M_r^m \ ; \ A_0 \cdot A_1 \cdots A_{m-1} = A_1 A_2 \cdots A_{m-1} A_0 = \ldots = p \cdot \mathrm{Id} \}$$

The morphism φ is smooth (Faltings, + many others).

In general, this does not seem to help much. But if $m = 2$,

also appears in moduli space of bundles on semi-stable curves (Faltings)

the special fiber of $N_{r,2}$ is

$$N \otimes \mathbb{F}_p = \{ (X, Y) \in M_r \times M_r \ ; \ XY = YX = 0 \}.$$

<u>Circular variety</u>, studied by Strickland and later by Mehta + Trivedi, and by Faltings.

They in particular proved that $N \otimes \mathbb{F}_p$ is <u>reduced</u>.

This implies that $N_{r,2}$ is flat – and this is used in

Theorem (Görtz): M_I is flat. The special fiber is reduced, its irreducible components are normal with rational singularities.

For $n \leq 5$, M_I is known to be Cohen-Macaulay.

Proof uses normality of Schubert varieties in affine flag variety. In fact, using this, one can circumvent the result of Strickland, etc.

3. The ramified unitary group.

Now assume p ramified in L. Change notation: Now $L/\mathbb{Q}_p$ ramified quadratic extension, ϖ uniformizer with $\overline{\varpi} = -\varpi$.

Let V = L-vector space of dim n, and

$$\phi : V \times V \longrightarrow L \qquad \text{non-deg. hermitian form.}$$

Let $$\langle v, w \rangle = \mathrm{Tr}_{L/\mathbb{Q}_p}(\varpi^{-1} \phi(v, w)) \qquad \text{alt. form.}$$

Let $e_1, \ldots, e_n$ basis of V s.t. $\phi(e_i, e_{n+1-j}) = \delta_{ij}$.

For $i = 0, \ldots, n-1$ put

$$\Lambda_i = \mathrm{span}_{O_L}\{\varpi^{-1} e_1, \ldots, \varpi^{-1} e_i, e_{i+1}, \ldots, e_n\}.$$

Extend to periodic lattice chain by $\Lambda_{i+kn} = \varpi^{-k} \Lambda_i$. Then

$$\hat{\Lambda}_j = \Lambda_{-j} \qquad (\text{for } \phi \text{ or } \langle\,,\rangle \text{ the same})$$

Fix $I=\{i_0<i_1<\dots<i_{m-1}\}\subset\{0,\dots,n-1\}$ with

$$i\in I,\ i\neq 0 \Rightarrow n-i\in I.$$

Also put $E=L$ if $r\neq s$, $E=\mathbb{Q}_p$ if $r=s$.

The <u>naive local model</u> is O_E-scheme repr. following moduli pb.

$M_\cdot^{naive}(S)=M_I^{naive}(S)=$ iso-classes of diagrams

$$\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \to & \Lambda_{i_1,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \xrightarrow{\varpi} & \Lambda_{i_0,S} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_{i_0} & \to & \mathcal{F}_{i_1} & \to & & \to & \mathcal{F}_{i_{m-1}} & \to & \mathcal{F}_{i_0}
\end{array}$$

Here $\mathcal{F}_{i_j}$ are loc. direct summands free of rank n, stable under O_L.

Following conditions are imposed:

- $\forall i\in I$ the composition

$$\mathcal{F}_i \hookrightarrow \Lambda_{i,S} \simeq \underset{\text{via } \hat{\Lambda}_i=\Lambda_{-i}}{\Lambda^*_{n-i}} \longrightarrow \mathcal{F}^*_{n-i} \quad \text{is zero}$$

If $0\in I$, want $\mathcal{F}_0$ tot. isotropic (i.e. in middle $\Lambda_{0,S}=\Lambda^*_{0,S}$).

- $\forall i\in I$,

<u>Kottwitz condition.</u> $\quad \mathrm{char}_{\varpi|\mathcal{F}_i}(T)=(T-\varpi)^r\cdot(T+\varpi)^s\in O_E[T]$

Theorem (Pappas): Let $I = \{0\}$ (i) If $|r-s| \geq 2$, then $M^{naive}_{\{0\}}$ is not flat, in fact dim (special fiber) > dim (generic fiber).

(ii) If $|r-s| \leq 1$, then $M^{naive}_{\{0\}}$ is flat if $n \leq 3$.

For $|r-s| \geq 2$ and $I = \{0\}$, Pappas proposed an additional condition on the action of ω on $\mathcal{F}_0$ $\left(\Lambda^{r+1}(\iota(\bar\omega)-\bar\omega) = 0 = \Lambda^{s+1}(\iota(\bar\omega)-\bar\omega) \right)$ and conjectures that the closed subscheme $M^{pappas}_{\{0\}}$ of $M^{naive}_{\{0\}}$ is flat. He can prove this if $r = n-1$, $s = 1$. Leads to following problem on matrix varieties: Consider the scheme

$$\{ X \in M_n \; ; \; {}^tX = X, \; \mathrm{char}_X(T) = T^n, \; X^2 = 0, \; \Lambda^{s+1} X = 0, \Lambda^{r+1} X = 0 \}.$$

Is this scheme reduced? In general have spin condition!

If $I = \{i, n-i\}$, $i \neq 0$, then are lead to the following matrix variety:

$$\left\{ (X_0, X_1) \in M_{2n} \times M_{2n} \; ; \; {}^tX_0 = -X_0, \; {}^tX_1 = X_1, \; X_0 = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}, \right.$$
$$\left. X_1 = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}, \; (X_1 X_0)^2 = p, \; + \text{signature cond. } (r,s) \right\}.$$

We know that this is not flat, we would like to understand the flat closure – does it have reduced special fiber etc. ?

4. <u>Other cases</u>

a) <u>Siegel moduli space (Görtz)</u>: M_I^{naive} is flat w. reduced fiber etc.

$I = \{r, 2n-r\}$ Leads to matrix variety

$$\{ X \in M_{2r} \; ; \; X^{ad} \cdot X = X \cdot X^{ad} = p \cdot I \} \qquad ad = \text{adj. w.r.t. sympl. } J.$$

Is flat, with <u>reduced</u> special fiber (uses a result of De Concini on coord.-ring of this scheme).

b) <u>Restriction of scalars</u> of Siegel or unramified unitary (Pappas, R) : M_I^{naive} is not flat, but flat closure M_I has all good properties. In addition,

$M_I \otimes_{\mathcal{O}_E} \mathcal{O}_K$ = twisted product of naive local models for unramified groups, hence have description of $M_I(k_{\bar{}})$.

c) <u>Hilbert-Blumenthal case</u>: This is the origin of all these questions. Rapo's mistake (discovered by Pappas) $\to$ Deligne-Pappas, Kottwitz-RZ

We understand now better (but still not completely) why these 2 solutions give the same answer.