Toronto, Nov. 01

<u>Talk: Converses to Mazur's inequality</u>

F - finite ext. of $\mathbb{Q}_p$

L = completion of F^{un}

$\sigma \in Aut(L/F)$ relative Frobenius

O_F, O_L rings of integers

$\pi \in O_F$ uniformizer

F-<u>isocrystal</u> = (N, φ), $\varphi: N \to N$ σ-linear bijective.

form a category. Classific. upto isomorphism (Dieudonné)

<u>Newton map</u>. $\{$ isocrystals of dim $n \}/\cong \longrightarrow (\mathbb{Q}^n)_+$; $(N, \varphi) \mapsto \nu(N, \varphi)$

<u>This map is injective, can charact. its image.</u>

Let (N, φ) F-isocrystal of dim. n. Let M be O_L-lattice in N. Associate <u>Hodge vector</u>

$$\mu(M) \in (\mathbb{Z}^n)_+.$$

Here $\mu(M) = (\mu_1, \dots, \mu_n) \Leftrightarrow \exists$ O_L-basis $e_1, \dots, e_n$ of M s.t. $\pi^{\mu_1} e_1, \dots, \pi^{\mu_n} e_n$ basis of $\varphi(M)$.

On $(\mathbb{Q}^n)_+$ usual dominance order.

<u>Mazur's inequality</u>: $\mu(M) \geq \nu(N, \varphi)$.

Theorem (Kottwitz, R): Let (N, φ) of dim. n. Let $\mu \in (\mathbb{Z}^n)_+$ such that $\mu \geq \nu(N, \varphi)$. Then $\exists$ M in N with $\mu = \mu(M)$.

Also due to Fontaine, see the end.

Group theoretic description (previous relates to GL_n).

G = reductive gp / F.

$B(G) = G(L) / \sigma$-conjugacy.

Get Newton map

$$\nu : B(G) \longrightarrow \mathfrak{a}_+$$

Here $\mathfrak{a}_+$ is defined as follows if G is quasi-split :

Let A max. split torus, $T = \mathrm{Cent}(A)$, $B \supset T$ Borel.

$\mathfrak{a} = X_*(A) \otimes \mathbb{Q}$, $\mathfrak{a}_+ = \mathfrak{a} \cap \overline{C}$.

This map is not injective in general, need also Kottwitz map

$$\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma = X^*(Z(\hat{G})^\Gamma).$$

Then (ν, κ) is injective, can charact. the image (Kottwitz)

Now let K_0 be an Iwahori subgroup of $G(L)$ defined over F. Then

$$K_0 \backslash G(L) / K_0 \simeq \widetilde{W} \quad \text{Iwahori Weyl gp},$$

hence get

$$\mathrm{inv}_K : G(L)/K_0 \times G(L)/K_0 \longrightarrow \widetilde{W}.$$

Here $\tilde{W}$ is defined as follows: S max. split torus in $G \otimes_F L$,

$T = \mathrm{Cent}(S)$, $N = \mathrm{Norm}(S)$. Assume S adapted to K_0.

Then

$$\tilde{W} = N(L) / N(L) \cap K_0 .$$

Have exact sequence which splits

inertia

$$1 \to X_*(T)_I \to \tilde{W} \to W_0 \to 1 ,$$

where $W_0 = N(L) / T(L)$ relative Weyl group

[Relation to affine Weyl group $W_a = X_*(S_{sc}) \rtimes W_0$:

$$1 \to W_a \to \tilde{W} \to X_*(T)_I / X_*(S_{sc}) \to 1 ,$$

which splits.]

Fix $b \in G(L)$ and $w \in \widetilde{W}$ $\rightsquigarrow$ assoc. affine "DL-variety"

$$X_{K_0}(b)_w = \{ g \in G(L)/K_0 \ ; \ \mathrm{inv}(b\sigma(g), g) = w \}.$$

More generally, let $K \supset K_0$ be a parahoric subgroup of $G(L)$ defined over F. Then

$$K \backslash G(L) / K \ \simeq \ \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K \ ,$$

$N(L) \cap K / N(L) \cap K_0$

Hence get

$$\mathrm{inv}_K : G(L)/K \times G(L)/K \longrightarrow \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K .$$

For $b \in G(L)$ and $w \in \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K$ $\rightsquigarrow$ $X_K(b)_w$

generalized DL-variety

Question: When is $X_K(b)_w \neq \emptyset$?

Special case: Let K be hyperspecial. Then G is quasisplit and splits over L and

$$\widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K = X_*(T) / W_0 .$$

Fix $b \in G(L)$ and $w \simeq \mu \in X_*(T) \cap \overline{C}$.

group-theor. version of Mazur's inequ.

K hyperspecial. Let $X_K(b)_\mu \neq \emptyset$. Then

(i) $\overline{\mu} \geq \nu(b)$ in $\mathfrak{a}_+$

(ii) $\kappa(b) = \mu^\natural$ in $\pi_1(G)_\Gamma$

$\Big\}$ $[b] \in B(G, \mu)$.

depends only on σ-conj. class and conj. class

Here $\overline{\mu}$ is the mean value of the orbit of μ under Γ.

Conjecture: Converse holds. True for GL_n (see above) and GSp_{2n}, and some more cases

(Iwahori case)

But in general have no idea (Dan Reuman)$\}$

<u>Situation changes drastically when form a certain finite union of affine DL-varieties.</u>

Fix conj-class μ of cocharacters of G. For a Borel subgroup $B \supset T$ defined over L (ex. by Steinberg) let $\mu_0 \in X_*(T)_{dom}$ be representative of $\{\mu\}$. The orbit under W_0 of the image of μ_0 in $X_*(T)_I$ is indep't of choice of B. Denote this orbit by

$$\Lambda = \Lambda(\{\mu\}) \subset X_*(T)_I \subset \tilde{W}.$$

Let <u>admissible subsets</u>

$$\mathrm{Adm}(\mu) = \{ w \in \tilde{W} \ ; \ w \leq \lambda \text{ for some } \lambda \in \Lambda \}$$

and $\mathrm{Adm}_K(\mu)$ = image of $\mathrm{Adm}(\mu)$ in $\tilde{W}^K \backslash \tilde{W} / \tilde{W}^K$

Haines / Ngo ← (finite subsets)

Put

$$X_K(b, \mu) = \bigcup_{w \in \mathrm{Adm}_K(\mu)} X_K(b)_w$$

Let μ be minuscule

<u>Conjecture</u>: (i) $X_K(b, \mu) \neq \emptyset \iff [b] \in B(G, \mu)$.

(ii) If $K' \supset K$ is another parahoric subgp defined over F, then

$X_K(b, \mu) \longrightarrow X_{K'}(b, \mu)$ is surjective.

Theorem (Kottwitz, R): The conjecture holds if $G = R_{F'/F}\, GL_n$ or $G = R_{F'/F}\, GSp_{2n}$, for some unramified extension F'/F.

Schlecht

The motivation for the set $X_K(b, \mu)$ comes from the problem of describing the set of points in a given isogeny class of a suitable integral model of a Shimura variety. Starting with $(\underline{G}, \{h\})$ Shimura datum and an open compact subgp $K = K^p \cdot K_p$

$$G = \underline{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p, \quad \mu = \{\mu_h\}, \quad K = K_p(L).$$

(ss) model $Sh(\underline{G}, h)_K$ flat over $O_{E(p)}$. Furthermore, to an isogeny class associate $b \in G(L)$ up to σ-conjugacy. Then the points in this isogeny class should be of the form

$$X^p \times X_K(b, \mu),$$

where $X^p \simeq \underline{G}(\mathbb{A}_f^p)/K^p$.

"Theorem" (Pappas, R): This description holds if $G = R_{F'/F}\, GL_n$ or $G = R_{F'/F}\, GSp_{2n}$, for some finite extension F'/F.

Important ingredients: a) Görtz's flatness theorems

b) Descr. of Haines / Ngo of $Adm(\mu)$.

Final remark: Mazur's inequality is not the only one. Since one of the topics of the conference is also p-adic uniformization, let me state another inequality and its converse.

Let (N, φ) be an isocrystal of dimension n. Let $\mathcal{F}^{\bullet}$ be a $\mathbb{Z}$-filtration of N of type $\mu = \mu(\mathcal{F}^{\bullet}) \in (\mathbb{Z}^n)_+$, i.e.

$$\dim gr_i^{\mathcal{F}^{\bullet}}(N) = \# \{ j ; \mu_j = i \}.$$

Fontaine's inequality: If $\mathcal{F}^{\bullet}$ is a (weakly) adm. filtr., then

$$\mu(\mathcal{F}^{\bullet}) \geq \nu(N, \varphi).$$

Converse: Let G be quasisplit $/ F$, let $\mu \in X_*(T) \cap \bar{C}$, $\rightsquigarrow$ $\mathbb{F}(G, \mu) / E$ corresp. partial flag var. over E.

Let $b \in G(L)$ $\rightsquigarrow$

$$\mathbb{F}(G, \mu)^{adm} \subset \mathbb{F}(G, \mu) \otimes_E \breve{E}, \quad \breve{E} = EL$$

open rigid-analytic subvariety (p-adic period domain)

Theorem (Fontaine, R): $\mathbb{F}(G, \mu)^{adm} \neq \emptyset \iff \bar{\mu} \geq \nu(b)$ in $\mathfrak{A}_+$.