Triest, Aug. 00

Talk: Flag varieties of loop groups and local models of Shimura varieties.

Motivation: As shown in Wedhorn's talk, the proof by Harris / Taylor is based on study of integral model of the Shimura variety assoc. to a unitary group $U(\tilde{F}/F, \Phi)$ where F tot. real, $\tilde{F}/F$ quadratic,

$$\mathrm{sgn}(\Phi_v) = \begin{cases} (n-1, 1) & v = v_0 \\ & \qquad v \mid \infty \\ (n, 0) & v \neq v_0 \end{cases}.$$

at a prime $\mathfrak{p}$ of the reflex field $E = \tilde{F}$ which of degree 1 over F (split case).

→ vanishing cycles.

We want to define integral models for more general Shimura varieties for unitary groups in case of split primes. These are moduli spaces of some abel. var. + polariz. (+ level str. outs. p). Via deep theorems (Serre / Tate, Groth / Messing) this can be reduced to a linear problem (Trivial if F unramif. at p (smooth).). From now on, we will only talk about this linearized problem (in the totally ramified case), may forget about the origin. We will define some projective schemes which have same singularities as Shimura varieties and their integral models (local models)

joint w. G. Pappas.

Naive local models:

$\mathbb{Q}_p \hat{=}$ F_0 = complete discr. valued field, perf. res.-field, $\mathcal{O}_{F_0}$

F/F_0 = totally ramified of degree e, $\mathcal{O}_F$; π Eisenstein $Q(T) = T^e + \sum b_i T^i$

V = F-vector space of dimension d

Λ = $\mathcal{O}_F$ - lattice in V.

Signature $\underline{r} = (r_\varphi)_\varphi$. Here $\varphi : F \longrightarrow F_0^{sep}$,

$$0 \leq r_\varphi \leq d \quad , \forall \varphi.$$

Let $r = \sum r_\varphi$. Define E = local Sh.-field (reflex field)

$$Gal(F_0^{sep}/E) = \{ \sigma \in Gal(F_0^{sep}/F_0);\ r_{\sigma\varphi} = r_\varphi \ , \forall \varphi \}.$$

Define projective scheme $M = M^{naive} = M(\Lambda, \underline{r})/\mathcal{O}_E$ by

Naive local model

$$M(S) = \{ \mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S;\ \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \text{- submodule},$$

loc. on S a direct summand as $\mathcal{O}_S$-module,

$$\text{char}(a \mid \mathcal{F}) \equiv \prod_\varphi (T - \varphi(a))^{r_\varphi} \}.$$

↑ Kottwitz condition $\Rightarrow rk_{\mathcal{O}_S} \mathcal{F} = r$

Analysis of generic fibre of M: Let K = Galois closure of F/F_0.

Then $K \supset E$ and

$$F \otimes_{F_0} K = \bigoplus_\varphi K$$

$$V \otimes_{F_0} K = \bigoplus V_\varphi \longleftarrow \text{all dim} = d$$

$$M \otimes K = \{ (\mathcal{F}_\varphi);\ \mathcal{F}_\varphi \in Grass_{r_\varphi}(V_\varphi) \} = \prod Grass_{r_\varphi}(V_\varphi)$$

<u>Analysis of special fiber of</u> M: Let k = residue field of O_E.

Let $W = \Lambda \otimes_{O_{F_0}} k$: a k-vs of dim. de

+ nilpotent operator $\Pi = \pi \otimes id_k$.

$$M \otimes_{O_E} k = \{ \mathcal{F} \in Grass_r(W) ; \ \mathcal{F} \ \Pi\text{-stable and} \ char(\Pi \mid \mathcal{F}) \equiv T^r \}.$$

<u>Question: When is M flat over</u> Spec O_E ? <u>And if not, can one define a natural closed subscheme</u> M^{loc} <u>of</u> M <u>which is flat</u> ?

A first check on flatness is dimension criterion. Obviously

$$\dim M \otimes_{O_E} K = \sum_\varphi r_\varphi (d - r_\varphi).$$

Π indeterminate

<u>Affine Grassmannian for</u> GL_d: Consider functor on k-algebras

$\widetilde{Grass}_k$: $R \longmapsto \{$ projective $R[[\Pi]]$-module $\mathcal{L}$ in $R((\Pi))^d$ s.t. $\exists n > 0$

$$\Pi^n R[[\Pi]]^d \subset \mathcal{L} \subset \Pi^{-n} R[[\Pi]]^d \}.$$

Ind-scheme, w. action of loop group $\tilde{\mathcal{G}}$ on it,

$$\tilde{\mathcal{G}}(R) = GL_d(R((\Pi))),$$

and in fact

$$\widetilde{Grass}_k = \tilde{\mathcal{G}} / \tilde{\mathcal{G}}^+.$$

Have action of $\tilde{\mathcal{G}}^+$ on $\widetilde{Grass}_k$ and the orbits are parametrized by dominant coweights of GL_d, (elementary)

$$\underline{s} = (s_1 \geq \dots \geq s_d) \mapsto \mathcal{O}_{\underline{s}}$$

And $\dim \mathcal{O}_{\underline{s}} = \langle \underline{s}, 2\varrho \rangle$, $\mathcal{O}_{\underline{s}'} \subset \overline{\mathcal{O}_{\underline{s}}}$ iff $\underline{s}' \leq \underline{s}$.

<u>Relation to local model</u>: Fix an isomorphism $W = \Lambda \otimes_{\mathcal{O}_{F_0}} k = (k[\![\Pi]\!]/\Pi^e)^d$. Then obtain closed embedding

$$i : M \otimes_{\mathcal{O}_E} k \hookrightarrow \widetilde{\mathrm{Grass}}_k$$

On k-rat. points: $\mathcal{F} \subset W \longmapsto \tilde{\mathcal{F}} \subset k[\![\Pi]\!]^d$.

In this way, get (Jordan type of $\Pi \mid \mathcal{F}$)

$$(M \otimes_{\mathcal{O}_E} k)_{red} = \bigcup_{\underline{s} \in \mathcal{S}^0(r,e,d)} \mathcal{O}_{\underline{s}}$$

$\mathcal{S}^0(r,e,d) = \{ \underline{s} ;\ e \geq s_1 \geq \dots \geq s_d \geq 0 ,\ \sum s_i = r \}$.

But $\underline{s}_{max} = (e, \dots, e, f, 0, \dots, 0)$ $r = c \cdot e + f$, $0 \leq f < e$

is unique maximal element in $\mathcal{S}^0(r,e,d)$. Hence

$$\dim M \otimes_{\mathcal{O}_E} k = \dim \mathcal{O}_{\underline{s}_{max}} = \langle \underline{s}_{max}, 2\varrho \rangle .$$

example: $r = (1,0,\dots,0) \Rightarrow$ smooth.

<u>Elementary calculation now shows</u>:

- If all r_φ differ by at most one, then
$$\dim M \otimes_{\mathcal{O}_E} K = \dim M \otimes_{\mathcal{O}_E} k$$
- In all other cases $<$, hence M not flat.

<u>Conjecture</u>: In the first case, M is flat.

[illegible]

Theorem 1: Let $e = 2$. If $r_{\varphi_1} = r_{\varphi_2}$, then M is flat.

If $r_{\varphi_1} > r_{\varphi_2}$, identify E with F via φ_1. Let

$$M^{loc}(S) = \{ \mathcal{F} \in M(S),\ \Lambda^{r_2+1}(\pi - \varphi_1(\pi)\,\mathrm{Id} \mid \mathcal{F}) = 0 \}.$$

Then M^{loc} is a closed subscheme of M which is flat$/O_E$.

Theorem 2: Let M^{loc} = scheme-theoretic closure of $M \otimes_{O_E} E$ in M.

Then

(i) M^{loc} is normal and C-M

(ii) The special fiber $M^{loc} \otimes_{O_E} k$ is reduced, normal w. rational sing.

and

$$M^{loc} \otimes_{O_E} k = \bigcup_{\underline{s} \leq \underline{r}^\vee} O_{\underline{s}}$$

$\underline{t} = \underline{r}^\vee$ via $t_1 = \#\{\varphi;\ r_\varphi \geq 1\}$, $t_2 = \#\{\varphi;\ r_\varphi \geq 2\}$, ...

→ But no reasonable functor definition in general for $e > 2$!

The proofs are based on the following fundamental but very simple observation.

Let $N = \{ A \in Mat_{r \times r};\ \mathrm{char}(A) = \prod (T - \varphi(\pi))^{r_\varphi},\ Q(A) = 0 \}$

Then get ← choose a basis of $\mathcal{F}$

$$\begin{array}{ccc} & \tilde{M} & \\ {}^{GL_r\text{-torsor}}\swarrow^{\pi} & & \searrow^{\varphi} \\ M & & N \end{array}$$

Equivariant w.r.t. $GL_r \times GL_d(k[[\Pi]]/\Pi^e)$.

Theorem 3: The morphism φ is smooth of relative dim. rd.
This morphism relates us to nilpotent variety, since
But

$$N \otimes_{\mathcal{O}_E} k = \{ A \in \mathrm{Mat}_{r\times r} ;\ \mathrm{char}(A) \equiv T^r,\ A^e = 0 \},$$

which is a nilpotent orbit closure (up to nilpotents) and

- (Mehta, van der Kallen) : normal + CM
- Kostant: reduced, if $r \leq e$.
- (Weyman) : reduced if $e = 2$ or char $k = 0$. This implies the final conjecture under same hypothesis.
- (Strickland) : if $e = 2$ and add condition $\Lambda^{\frac{r}{2}+1} A = 0$, then reduced.

Further remarks:

(i) From Theorem 3 can also prove that all Schubert varieties in the affine Grassmannian for GL_d are smoothly equiv. to nilpotent orbit closure for GL_r, some r. This gives a ~~proof~~ of converse to result of Lusztig. In particular, these Schubert varieties are normal w. rational singularities.

Starting with this result, Görtz has proved the corresp. facts for all affine partial flag varieties for GL_d.

Lately, Faltings has extended this (with a different proof) to all G with G_{der} 1-connected).

(ii) Have a Demazure type resolution

$$\pi : \mathcal{M} \longrightarrow M^{loc} \otimes_{\mathcal{O}_E} \mathcal{O}_K$$

where $\pi \otimes_{O_K} K$ is isomorphism and $\mathcal{M}$ is smooth. Using it, one can calculate the vanishing cycles for $H^{loc} \otimes_{O_E} O_K / O_K$

<u>Definition of $\mathcal{M}$:</u> Let $\varphi_1, \dots, \varphi_e$ the embeddings and let $a_i = \varphi_i(\pi)$. Then

$$\mathcal{M}(S) = \{ (0) \subset \mathcal{F}_e \subset \mathcal{F}_{e-1} \dots \subset \mathcal{F}_e = \mathcal{F} \subset \Lambda \otimes_{O_{F_0}} O_S \, ;$$

$\mathcal{F} \in M(S)$, $\mathcal{F}_k$ loc. on S direct summand of rank $\sum_{i=1}^{k} r_i$ s.t. $(\pi - a_k \, Id)\, \mathcal{F}_{k-1} \subset \mathcal{F}_k$, $k = 1, \dots, e\}$