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Talk: Good and semi-stable reduction of Shimura varieties

joint w. X. He, G. Pappas. < Problem: GL_2 two questions.

§ 1 Local models.

Def.: Let F local field, fix alg. closure $\bar F$. A LM-triple over F = triple $(G, \{\mu\}, K)$, where

- G reductive gp $/F$
- $\{\mu\}$ conj. class of cochar. over $\bar F$, minuscule.
- K = parahoric subgp $\breve K$ of $G(\breve F)$ defined over F.

Homo / Iso of LM-triples. + base change $F \to F' \subset \bar F$.

To $(G, \mu, K) \mapsto \mathcal{G} = \mathcal{G}_K$ over O_F, $E(G, \{\mu\}) \subset \bar F$, $k = \bar k_E$.

Throughout always assume that G splits over tamely ramified ext. of F.

Also associate a discrete invariant, as follows.

Def.: An enhanced Tits system = triple $(\tilde\Delta, \{\lambda\}, \tilde K)$, where

- $\tilde\Delta$ = local Dynkin diagram

Let W_a = assoc. affine Weyl gp (Coxeter gp on set $\tilde S$ of vertices), W_0 = assoc. finite Weyl gp, $\tilde W$ = assoc. extended Weyl gp with translation subgp X_*.

- $\{\lambda\}$ = W_0-conj. class of elts in X_*
- $\tilde K$ = non-empty subset of $\tilde S$.

<u>Enhanced multi-Tits system</u> = product of finitely many enh. Tits systems.

Let $(G, \{\mu\}, K)$ LM-triple. Associate as follows enhanced multi-Tits syst.

$$G_{ad} \otimes_F \breve{F} = \prod_i \breve{G}_i$$

where $\breve{G}_i$ is $\breve{F}$-simple adjoint. For each i, let

$\tilde{\Delta}_i$ = local Dynkin diagram of $\breve{G}_i$

$\{\lambda_i\}$ = image of $\{\mu_i\}$ in $X_*(\breve{T}_i)_{\Gamma_0} = X_{i*}$.

$\tilde{K}_i$ = conj. class of $\breve{\mathcal{G}}_i$.

* ↓ Let $(G, \{\mu\}, K)$ LM-triple over F.

<u>Pappas-Zhu theory</u>: Let $F/\mathbb{Q}_p$. The theory depends on choice of $\pi \in O_F$ and a reductive gp scheme $\underline{G}$ over $O_F[u^\pm]$ such that

$$\underline{G} \otimes_{O_F[u^\pm],\, u \mapsto \pi} F = G$$

(<u>spreading out</u>). Let $\underline{\breve{G}} = \underline{G} \otimes_{O_F[u^\pm]} O_{\breve{F}}[u^\pm]$. Then PZ construct

- spreading out $\underline{\mathcal{G}}$ of $\mathcal{G}$ over $O_F[u]$, ~~s.th.~~ of $\mathcal{G}$ s.t.

$$G' = \underline{G} \otimes_{O_F[u^\pm]} k_F((u)) \quad \text{reductive gp}$$

$$\mathcal{G}' = \underline{\mathcal{G}} \otimes_{O_F[u]} k_F[[u]] \quad \text{parahoric gp scheme},$$

with canonical identific.

$$\mathcal{G} \otimes_{O_F} k = \mathcal{G}' \otimes_{k_F[[u]]} k.$$

- tori $\underline{\breve{A}} \subset \underline{\breve{T}} \subset \underline{\breve{G}}$ which induce max split/max. in every fiber.

and hence identifications

$$W_0(\breve G_{ad},\breve T_{ad}) = W_0(\breve G'_{ad},\breve T'_{ad}), \qquad \tilde W(\breve G_{ad},\breve T_{ad}) = \tilde W(\breve G'_{ad},\breve T'_{ad}),$$

$$\{\mu\} = \{\mu'\}, \quad \{\lambda\} = \{\lambda'\}, \quad \mathrm{Adm}_{\breve K}(\{\mu\}) = \mathrm{Adm}'_{\breve K'}(\{\mu'\}).$$

loc. multi-Tits of $(G,\{\mu\},K)$ = loc. multi-Tits of $(G',\{\mu'\},K')$.

Let $\mathcal F' = L\breve G' / L^+\breve{\mathcal G}'$, and

$$\mathcal A = \bigcup_{w\in \mathrm{Adm}'} S'_w \subset \mathcal F'$$

__Note__: Since $\{\mu\}$ minuscule, action of $L^+\breve{\mathcal G}'$ on $\mathcal A$ factors through $\mathcal G'\otimes k \simeq \mathcal G\otimes k$

__Fact__: The k-scheme with $\mathcal G\otimes k$-action $\mathcal A$ is, up to iso, __indep't__ of auxiliary choices: $\mathcal A(G,\{\mu\},K)$

__Definition__: A __local model__ for LM-triple $(G,\{\mu\},K)$ is a projective flat O_E-scheme $\mathbb M^{\mathrm{loc}}(G,\{\mu\},K)$ w. action by $\mathcal G\otimes_{O_F}O_E$ s.t.

a) generic fiber iso to $X_{\{\mu\}}$ with its G_E-action

b) geo. special fiber is reduced and isom. to $\mathcal A(G,\{\mu\},K)$, with its $\mathcal G\otimes k$-action.

__Conjecture__: There ex. a local model, unique up to isom.

PZ: Can construct (a class of) local models $\mathbb M^{\mathrm{loc}}(G,\{\mu\},K)$ w. foll. properties.

* <u>PZ-local models</u>: Let $F/\mathbb{Q}_p$, let $(G, \{\mu\}, K)$ LM-triple over F.

Then PZ associate $\mathbb{M}^{loc}(G, \{\mu\}, K)$ over $\mathcal{O}_E$ with $\mathcal{G}_{\mathcal{O}_E}$-action which is projective + flat +

- generic fiber iso. to $X_{\{\mu\}}$ with its G_E-action
- geom. special fiber reduced + isomorphic to $\mathcal{A}(G, \{\mu\}, K)$ with $\mathcal{G}\otimes k$-action

Here $\mathcal{A}(G, \{\mu\}, K)$ = admissible locus in a partial affine flag variety $\mathcal{F}'$ over k, assoc. to $(G', \{\mu'\}, K')$ over $k_F((u))$, with same enhanced multi-Tits system as $(G, \{\mu\}, K)$.

$$\mathcal{A}(G, \{\mu\}, K) = \bigcup_{w' \in \mathrm{Adm}_{K'}(\{\mu'\})} S_{w'} .$$

<u>Remarks</u>: (i) The construction of $\mathbb{M}^{loc}$ depends on auxiliary choices – but result should be independent. Hope: Two properties above characterize $\mathbb{M}^{loc}$.

(ii) see (ii), p. 4.

(iii) From now on we do, as if this independence is known.

(i) If K hs, then "any one" $\mathbb{M}^{loc}$ is smooth

(ii) If F'/F unramified, then

$$\mathbb{M}^{loc} \otimes_{O_E} O_{E'} \quad \text{is "one of"} \quad \mathbb{M}^{loc}(G \otimes_F F', \{\mu'\}, K')$$

(iii) product.

(iv) If $(G, \mu, K) \longrightarrow (G', \mu', K')$ s.t. $G \to G'$ central ext.,

then

$$\mathbb{M}^{loc}(G', \mu', K') \otimes_{O_{E'}} O_E \quad \text{is "one of"} \quad \mathbb{M}^{loc}(G, \{\mu\}, K).$$

<u>Remarks</u>: (i) $\mathbb{M}^{loc}(G, \{\mu\}, K)$ coincides with $M_{\underline{\mathcal{G}}, \{\mu\}}$ in [PZ],

(*) if $p \nmid \pi_1(G_{der})$ — and maybe always.

(ii) Assume G adjoint + classical, and that

$$G \otimes_F \breve{F} = \prod \breve{G}_i \quad ,$$

where each $\breve{G}_i$ absolutely simple. Assume that if $(\breve{G}_i, \{\mu_i\})$ of type D^H, then $p > 2$. Then $\mathbb{M}^{loc}(G, \{\mu\}, K)$ is a Scholze local model. In particular, then uniqueness via Scholze characterization — but does not help to prove conjecture.

§2 Shimura varieties.

Let (G, X) be Shimura datum, let p. Assume that $G = G \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is tamely ramified. Let $\mathbb{K} = \mathbb{K}^p . K$, where $K \subset G(\mathbb{Q}_p)$ parahoric.
Fix $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, let $E = \mathbb{E}_{\wp}$. Then E = reflex field of $(G, \{\mu\})$. Let $Sh(G, X)_{\mathbb{K}}$ can. model over $\mathbb{E}$, resp. over E.

Theorem: a) Assume (G, X) of Hodge type. Then ex. model $\mathcal{S}(G, X)_{\mathbb{K}}$ over O_E which admits a local model diagram

$$
\begin{tikzcd}
 & \tilde{\mathcal{S}}(G, X)_{\mathbb{K}} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\varphi}"] & \\
\mathcal{S}(G, X)_{\mathbb{K}} & & \mathbb{M}^{loc}(G, \{\mu\}, K)
\end{tikzcd}
$$

in which $\tilde{\varphi}$ is surjective. In particular, $\forall$ closed point x of $\mathcal{S}(G, X)_{\mathbb{K}}$, ex. closed point y of $\mathbb{M}^{loc}$ s.t. strict henzel. of x and y are isomorphic.

b) If (G, X) only of abelian type, and $A \otimes_{\mathbb{Q}} \mathbb{R} = A_{\mathbb{R}}^{\vee}$ (Serre cond.), then last statement still true.

Conjecture: a) should always hold if Serre cond. satisfied.

Corollary: In sit. of b), if $\mathbb{M}^{loc}$ good/semi-st. reduction, then also $\mathcal{S}(G, X)_{\mathbb{K}}$. In sit. of a), the converse also holds.
If Conj. true then $\mathcal{S} \Leftrightarrow \mathbb{M}^{loc}$.

§ 3 Good reduction.

Theorem 1: Let $(G, \{\mu\}, K)$ LM-triple s.t. G_{ad} F-simple. Assume

$$G_{ad} \otimes_F \breve{F} = \prod \breve{G}_i ,$$

where each factor $\breve{G}_i$ is abs. simple. Then $\mathbb{M}^{loc}(G, \{\mu\}, K)$ has good reduction if and only if either K is hs or (G, μ, K) is of exotic good reduction type.

Typical exotic g.r.t. (general case obtained via $Res_{F'/F}$, F'/F unramif.)

Let $\tilde{F}/F$ ramif. quadr., let $G = U(V)$, let $\{\mu\} = (1, 0, \ldots, 0)$, let $K = Stab(\Lambda)$, where $\Lambda = \begin{cases} \pi\text{-mod. if } \dim V \text{ even} \\ \text{almost } \pi\text{-mod if } \dim V \text{ odd.} \end{cases}$

§ 4 Semi-stable reduction.

Lemma: Let $(G_1, \{\mu_1\}, K_1)$ and $(G_2, \{\mu_2\}, K_2)$ be two LM-triples/F s.t. G_i adjoint + absol. simple. Then they define same enhanced Tits system if and only if they become iso. over an unramified extension of F.

Hand out.

§ 5 About proofs.

Proof of Theorem 1: Direct implication see Arzdorf / Richarz.

Converse: 3 steps

1) Enumerate all cases where $\mathrm{Adm}(K,\mu)$ one extreme element.

2.) Among those, eliminate all cases where special fiber not rationally smooth ($P = \sum_{w \leq w_{max}} t^{l(w)}$ symmetry).

3) Turns out that in all remaining cases, K special maximal parah. $L\check{G}'/L^+\mathcal{G}'$.

Theorem (Haines / Richarz): Consider $\mathcal{F}' =$ loop gp Grassmannian corresp. to special max parah. Then any Schubert variety S_w is singular along its boundary, unless $(\mathcal{F}', S_w)$ exotic good reduction case.

Remarks: (i) If $\check{G}'$ split and char $k = 0$, due to Evens / Mirkovic, resp. Malkin / Ostrik / Vybornov. Diplomarbeit Müller bei Görtz.

(ii) Need only a special case – much simpler.

Proof of Theorem 2: Direct implication: linear algebra.

Converse: 4 steps

1) semi-stable red $\Rightarrow$ (CCP): $\# \{$extreme elts in $\mathrm{Adm}(K,\mu)\} \leq \# \tilde{K}$.

Enumerate all cases where CCP satisfied.

2.) Eliminate all cases where int. of irred. comp. not rationally smooth

3) Eliminate all cases when tired comp. rationally smooth, but not smooth.

<u>Surprise</u>: All remaining cases are semi-stable.

$\ast$ Summary of PZ-theory: Let $F/\mathbb{Q}_p$. Let $(G, \{\mu\}, K)$ over F.

Theory depends on $\pi \in F$ and spreading out $\underline{\mathcal{G}}$ over $O_F[u]$:

$$\underline{\mathcal{G}} \otimes_{O_F[u],\, u \mapsto \pi} O_F = \mathcal{G}, \qquad \underline{\mathcal{G}} \otimes_{O_F[u],\, \pi \mapsto 0} k_F[\![u]\!] = \mathcal{G}' \text{ over } k_F[\![u]\!].$$

$$\text{multi-Tits}(G, \{\mu\}, K) = \text{multi-Tits}(G', \{\mu'\}, K').$$

Furthermore, let $\mathcal{F}' = L\breve{G}' / L^+\breve{\mathcal{G}}'$ affine flag over $/k$, and

$$\mathcal{A} = \bigcup_{w' \in \mathrm{Adm}_{\breve{K}'}(\{\mu'\})} S'_w \qquad L^+\breve{\mathcal{G}}' \text{ acts}$$

Since $\{\mu\}$ <u>minuscule</u>, action of $L^+\breve{\mathcal{G}}'$ factors through

$$\mathcal{G}' \otimes k = \mathcal{G} \otimes k.$$

<u>Fact</u>: The k-scheme $\mathcal{A}$ with $\mathcal{G}\otimes k$-action indep't of all choices.