Münster, April 18

Talk: Good + semi-stable reduction of Shimura varieties

Fix p

(1) <u>Example</u>: Modular curve $\mathcal{M}_{K^p \cdot K}$, $K = K_p = \begin{cases} GL_2(\mathbb{Z}_p) \\ \Gamma_0(p) \end{cases}$

$$\mathcal{O}^{sh} \simeq (W(\mathbb{F}_p)[X,Y]/XY-p)^{sh}$$

(2). <u>Problem</u>: a) some integr. model

b) a chosen class of int. models.

Joint w. X. He, G. Pappas. <u>2 anecdotes</u>

(3) <u>Class of Kisin-Pappas</u>: Start with Shimura datum $(\mathbf{G}, X)$ of abelian type s.t. $G = \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ splits over tamely ramif. ext.

Let $\mathbb{K} = K^p \cdot K$, where $K = K_p$ = parahoric. Then get

$$\mathcal{S}(\mathbf{G}, X)_{\mathbb{K}} = \text{good model over } \mathcal{O}_E, \quad E = E_{\mathfrak{p}} \; (\text{any } \mathfrak{p}/p).$$

<u>Explanations</u>:

a) abelian type $\supset$ Hodge type $\supset$ PEL-type, i.e. "G of classical type"

b) parahoric: for $G = GL_n$, stabilizer of periodic lattice chain

$$\Lambda_{i_1} \subset \Lambda_{i_2} \subset \ldots \subset \Lambda_{i_t} \subset p^{-1}\Lambda_{i_1}$$

std.

for $G = GSp_{2n}$, Λ_0 selfdual.

c) In general have no charact. But for K hyperspec. have one.

h.s. $\begin{cases} GL_n: & 1 \text{ lattice} \\ GSp_{2n}: & 1 \text{ selfdual lattice.} \end{cases}$

d) Singularities modeled by corresp. LM (linearization of problem).

(4.) Definition: Let $F/\mathbb{Q}_p$. An LM-triple $= (G, \{\mu\}, \mathcal{G})$ where

G reductive $/F$, $\{\mu\}$ = minuscule conj. class of $\mu : \mathbb{G}_{m_{\bar F}} \to G_{\bar F}$, $\mathcal{G}$ = smooth gp scheme over O_F s.t. $K = \mathcal{G}(O_F)$ parahoric.

$\rightsquigarrow$ $E \subset \bar F$.

Pappas-Zhu: Assume G tame. They construct $\mathbb{M}^{loc} = \mathbb{M}^{loc}(G, \{\mu\}, \mathcal{G}) =$ flat projective O_E-scheme ~~st~~ with $\mathcal{G} \otimes_{O_F} O_E$-action

generic fiber $= X_{\{\mu\}}$ as G_E-scheme

geom. special fiber $= \mathcal{A}(G, \{\mu\}, \mathcal{G})$ as $\mathcal{G} \otimes k$-schemes,

closed union of Schubert var. inside loop group flag var. $LG' / L^+ \mathcal{G}'$

Conjecture: These 2 properties characterize $\mathbb{M}^{loc}$.

Example: Let $F = \mathbb{Q}_p$, $G = GL_n$, $\{\mu\} = (1^{(r)}, 0^{(n-r)})$, $\mathcal{G} = \mathrm{Stab}(\overset{\text{std.}}{\Lambda_0 \subset \Lambda_1 \subset \cdots})$.

Flag variety is $L\,GL_{n\,k((t))} / L^+ \mathcal{G}'$, $\mathcal{G}' = \mathrm{Stab}(\Lambda_0' \subset \Lambda_1' \subset \cdots)$

Iwahori-Weyl gp of $LG' = \mathbb{Z}^n \rtimes S_n = \tilde W$

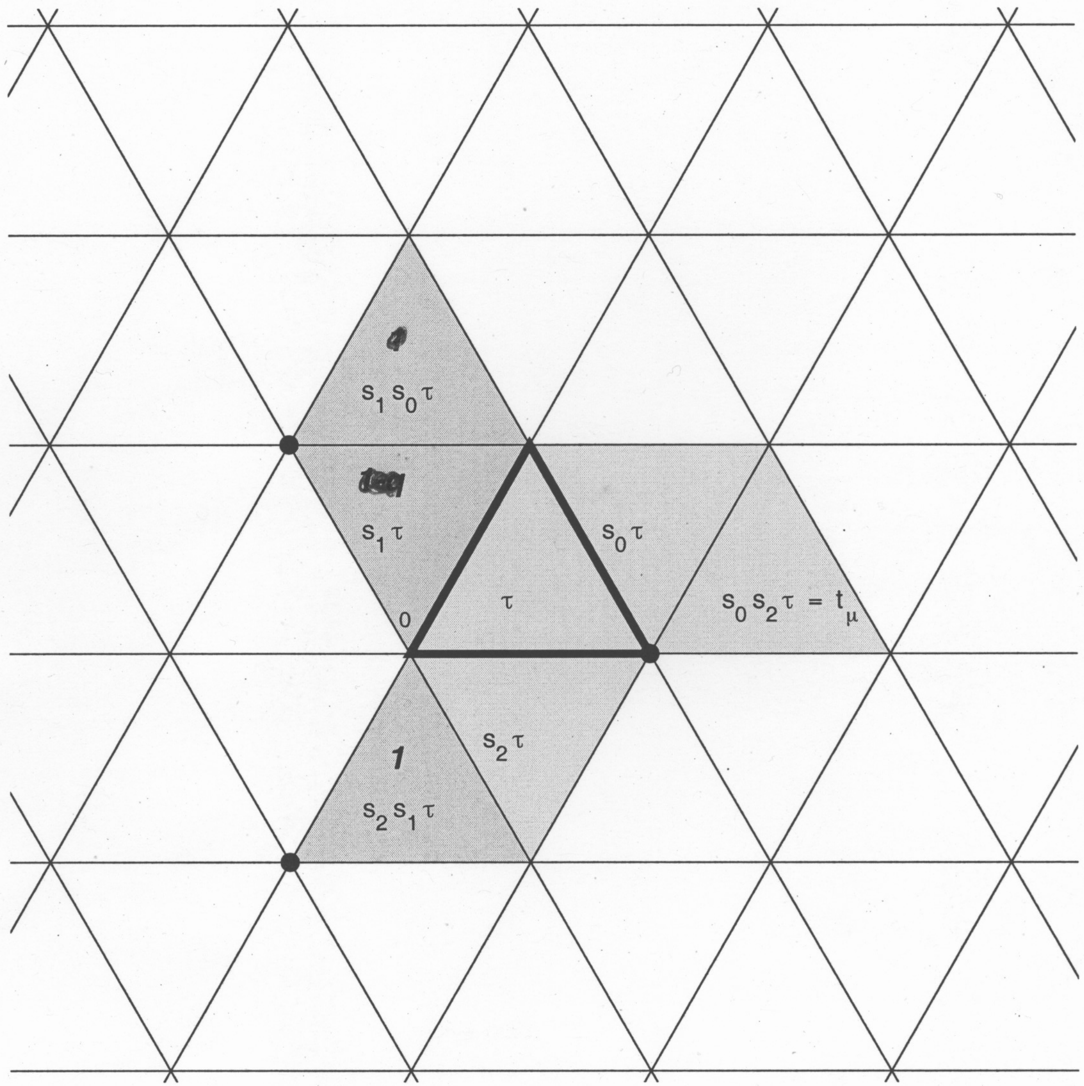
$\mathrm{Adm}(\mu) = \{ w \in \tilde W \mid w \leq t_{s(\mu)}$, some $s \in S_n \}$.

$$\mathcal{A}(G, \{\mu\}, \mathcal{G}) = \bigcup_{w \in \mathrm{Adm}(\mu)} S_w$$

$\mathbb{M}^{loc}$ Represents on $(Sch/\mathbb{Z}_p)$: $S \mapsto$

$$\begin{matrix} \Lambda_{0,S} & \to & \Lambda_{1,S} & \to & \cdots & \to & \Lambda_{n-1,S} & \to & \Lambda_{0,S} \\ \cup & & \cup & & & & \cup & & \cup \\ \mathcal{F}_0 & \to & \mathcal{F}_1 & & \cdots & \to & \mathcal{F}_{n-1} & \to & \mathcal{F}_0 \end{matrix}$$

(Görtz) $\quad \mathrm{rg}\, \mathcal{F}_i = r$.


s₁ s₀ τ
s₁ τ
0
τ
s₀ τ
s₀ s₂ τ = t_μ
1
s₂ τ
s₂ s₁ τ

<u>Modelling means</u>: Form LM-triple $G, \{\mu\}, \mathcal{G}$ over $\mathbb{Q}_p$: $G = \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $\{\mu\}$ from X, $\mathcal{G}$ from K_p.

Then $\forall$ closed point $x \in \mathcal{S}(\mathbf{G}, X)_{\mathbb{K}}$ ex $\exists$ closed point $y \in \mathbb{M}^{loc}$ s.t.

$$\mathcal{O}^{sh}_{\mathcal{S},x} \simeq \mathcal{O}^{sh}_{\mathbb{M}^{loc},y}.$$

Furthermore, if $(\mathbf{G}, X)$ of Hodge type, then also conversely.

<u>Example</u>: modular curve with $\Gamma_0(p)$, explains rat. double point.

Hence question of good / semi-stable red. on $\mathcal{S}$ transferred to $\mathbb{M}^{loc}$. From now on only LM.

<u>Remark</u>: LM should exist even without tameness assumption.

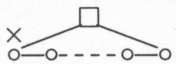
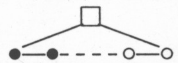
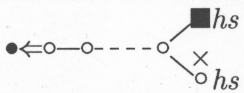
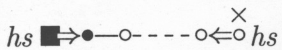
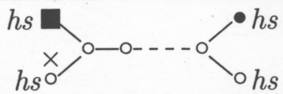
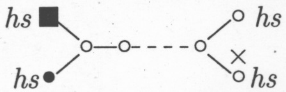
$\rightsquigarrow$ Scholze (Berkeley lectures).

Theorem. *Let $(G, \{\mu\}, K)$ be a LM-triple over F such that G splits over a tamely ramified extension of F. Assume that G_{ad} is F-simple and that in the product decomposition over $\breve{F}$,*

$$G_{\mathrm{ad}} \otimes_F \breve{F} = \prod \breve{G}_{\mathrm{ad},i}$$

each factor is absolutely simple. *Then the local model $\mathbb{M}^{\mathrm{loc}}(G, \{\mu\}, K)$ is smooth over* $\mathrm{Spec}\, O_E$ *if and only if K is hyperspecial or (G, μ, K) is an LM-triple of exotic good reduction type.*

Theorem. *Assume that G is adjoint and absolutely simple. The local model $\mathbb{M}^{\mathrm{loc}}(G, \{\mu\}, K)$ has semi-stable but not good reduction in precisely one of the following cases (we list the corresponding enhanced Tits data):*

Enhanced Tits datum	*Linear algebra datum*	*Discoverer*
 *All vertices are hyperspecial* $\sharp\tilde{K} \geq 2$	*Split* SL_n, $r = 1$ *arbitrary chain of lattices of length* ≥ 2	*Drinfeld*
 *All vertices are hyperspecial* μ *is any minuscule coweight*	*Split* SL_n *with* $n \geq 4$ r *arbitrary,* (Λ_0, Λ_1)	*Görtz*
	*Split* SO_{2n+1} *with* $n \geq 3$, $r = 1$, (Λ_0, Λ_n)	*new*
	*Split* Sp_{2n} *with* $n \geq 2$, $r = n, (\Lambda_0, \Lambda_1)$	*new*
	*Split* SO_{2n} *with* $n \geq 4$, $r = 1, (\Lambda_0, \Lambda_n)$	*Faltings*
	*Split* SO_{2n} *with* $n \geq 5$, $r = n, \Lambda_1$	*new*

In the diagrams above, if not specified, hyperspecial vertices are marked with an hs. In order to also show the coweight $\{\lambda\}$, a special vertex is specified (marked by a square)[1] *so that the extended affine Weyl group appears as a semi-direct product of W_0 and X_*. Then $\{\lambda\}$ is equal to the fundamental coweight of the vertex marked with $\times$. Finally, the subset $\tilde{K}$ is the set of vertices filled with black color.*

Note that there are some obvious overlaps between the first two rows.

[1] Note that the local Dynkin type C-BC_n does not occur here so that all special vertices are conjugate; hence this specification plays no role.

anecdote.

Exotic good reduction (Richarz)

Let F'/F ramif. quadr., let $G = U(V)$ unitary grp, $\{\mu\} = (1, 0, .., 0)$,

K = Stab of π-mod. / almost π-mod lattice in V.

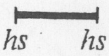
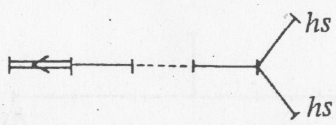
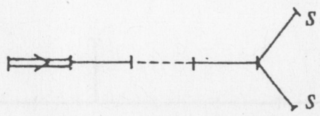
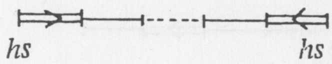
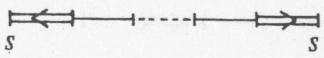
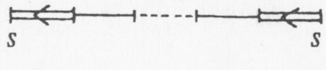
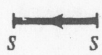
Only example I know where smoothness not by infinit criterion / homogeneity.

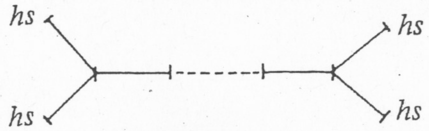
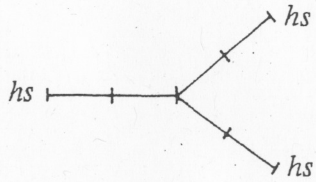
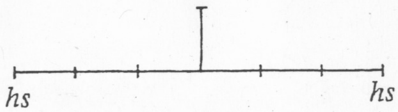
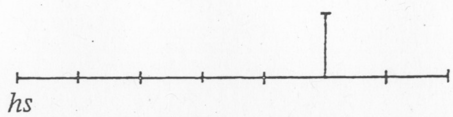
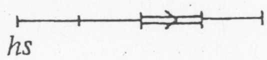
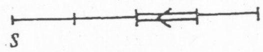
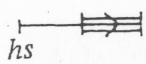
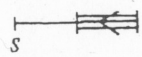
Closely related to smoothness of Schubert varieties in affine Grassmannian (Evens / Mirkovic, Haines / Richarz).

Under hyp. of Thm 2, ~~the~~ enhanced Tits systems of two LM equal $\Longleftrightarrow$ become isom. over unramif. ext. of F

Anecdote

4.2. *Residually split groups.*

Name	Local Dynkin diagram	Index [22]
A_n $(n \geqq 2)$	A cycle of length $n+1$ all vertices of which are hyperspecial	${}^1A^{(1)}_{n,n}$
A_1		${}^1A^{(1)}_{1,1}$
B_n $(n \geqq 3)$		$B_{n,n}$
B-C_n $(n \geqq 3)$		${}^2A^{(1)}_{2n-1,n}$
C_n $(n \geqq 2)$		$C^{(1)}_{n,n}$
C-B_n $(n \geqq 2)$		${}^2D^{(1)}_{n+1,n}$
C-BC_n $(n \geqq 2)$		${}^2A^{(1)}_{2n,n}$
C-BC_1		${}^2A^{(1)}_{2,1}$

Name	Local Dynkin diagram	Index [22]
D_n $(n \geqq 4)$		${}^1D^{(1)}_{n,n}$
E_6		${}^1E^0_{6,6}$
E_7		$E^0_{7,7}$
E_8		$E^0_{8,8}$
F_4		$F^0_{4,4}$
F^1_4		${}^2E^2_{6,4}$
G_2		$G_{2,2}$
G^1_2		${}^3D_{4,2}$ or ${}^6D_{4,2}$