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Talk: Local models of Shimura varieties in the ramified case

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Want: Integral models of Shimura var. with parahoric level str.

Typical example:

F = totally real, $\tilde{F}/F$ totally imag. quadr.

V = $\tilde{F}$-vector space, with $(,) : V \times V \to \mathbb{Q}$ non-deg. alt. s.t.

$$(ax, y) = (x, \bar{a} y), \quad a \in \tilde{F}.$$

$G = GU(V, (,))$. Let $h : \mathbb{C}^\times \to GU_{\mathbb{R}}$ with usual

Riemann conditions $\leadsto$ $Sh(G, h)_K$ (moduli space of ab. var.) $/ E$ = reflex field, $\subset \bar{\mathbb{Q}}$.

Fix p, impose foll. conditions: • $K = K^p . K_p$ with $K_p \subset G(\mathbb{Q}_p)$ parahoric

(For convenience of notation) → • p remains prime in F and is tot. ramified: $\mathfrak{p} = (p)$

• $\mathfrak{p}$ unramified in $\tilde{F}/F$.

To (G, h) is assoc. a conj-class $\{\mu\}$ of coch. of G

Fix $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. We want local model $M = M(G_{\mathbb{Q}_p}, \{\mu\})_{K_p}$ over $O_{E_\mathfrak{p}}$ (flat etc)

and smooth morphism of relative dim = dim G,

$$Sh(G, h)_K^{\sim} \longrightarrow [M / \mathcal{G}_{O_E}]$$

Here $Sh(G,h)_K^{\sim}$ is integral model of $Sh(G,h)_K \otimes_E E_\mathfrak{p}$ and $\mathcal{G} / \mathbb{Z}_p$

smooth group scheme (w. connected fibers) s.t. $\mathcal{G}(\mathbb{Z}_p) = K_p$.

New notation :

F_0 = complete discr. valued field, perfect residue field, $\mathcal{O}_{F_0}$

F/F_0 = totally ramified separable of degree e, $\mathcal{O}_F$

π uniformizer

V = F-vector space of dim d, $e_1, \dots, e_d$ basis.

Λ_i = $\mathrm{span}\{\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_d\}$ $i = 0, \dots, d$.

- part of a periodic lattice chain.

Let $I = \{i_0 < i_1 < \dots < i_{m-1}\}$ non-empty in $\{0, \dots, d-1\}$.

Let

$G = R_{F/F_0}(GL(V))$, $K = K_I$ = $\mathrm{Stab}\{\Lambda_i ; i \in I\} \subset G(F_0)$ parahoric subgp.

Also fix $(r_\varphi)_{\varphi \in \mathrm{Hom}_{F_0}(F, \overline{F}_0)}$ with $0 \le r_\varphi \le d$: $\hat{=}\{\mu\}$.

Local reflex field E defined by $\mathrm{Gal}(\overline{F}_0/E) = \{\sigma ; r_{\sigma\varphi} = r_\varphi, \forall\varphi\}$

Let k = residue field of $\mathcal{O}_E$.

We can now define the naive local model M^{naive} = $M^{\mathrm{naive}}(G, \{\mu\})_I$, a projective scheme over $\mathcal{O}_E$. It represents the following functor : $S \mapsto$ iso-classes of commut. diags.

explain

$$\begin{array}{ccccccccc} \Lambda_{i_0,S} & \to & \Lambda_{i_1,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \xrightarrow{\pi} & \Lambda_{i_0,S} \\ \cup & & \cup & & & & \cup & & \cup \\ \mathcal{F}_0 & \to & \mathcal{F}_1 & \to & \cdots & \to & \mathcal{F}_{m-1} & \to & \mathcal{F}_0 \end{array}$$

where $\mathcal{F}_i$ loc. free $\mathcal{O}_S$-submodules of rank $r = \sum r_\varphi$ which are loc. direct summands, and which are stable under action of $\mathcal{O}_F$ s.t.

keep on blackboard

$$\mathrm{char}(\iota(a); \mathcal{F}_i) = \prod_\varphi (T - \varphi(a))^{r_\varphi} \in \mathcal{O}_E[T], \ \forall a \in \mathcal{O}_F.$$

Let $\mathcal{G} = \underline{\mathrm{Aut}}(\Lambda_i ; i \in I)$: acts on $\mathcal{M}$.

Easy to see:

$$\mathcal{M}^{\text{naive}} \otimes_{\mathcal{O}_E} E \simeq \prod_\varphi \mathrm{Grass}_{r_\varphi, d - r_\varphi}$$

$$\mathcal{M}^{\text{naive}} \otimes_{\mathcal{O}_E} k \hookrightarrow \widetilde{\mathcal{F}}_{\underline{I}} = GL_d(k[[\Pi]])/\mathcal{P}_{\underline{I}}$$

closed subvariety, stable under $\mathcal{P}_{\underline{I}}$.

Thm 1 (Görtz): Let $e = 1$ (unramified case). Then $\mathcal{M}^{\text{naive}}$ is flat$/\mathcal{O}_E$ with reduced special fiber. All irreducible components of special fiber are normal, CM with rational singularities. At least if $d \leq 4$, then $\mathcal{M}$ is CM. Furthermore,

$$\mathcal{M}^{\text{naive}} \otimes_{\mathcal{O}_E} k = \bigcup_{\mathrm{Adm}_{\underline{I}}(\mu)} \mathcal{O}_w$$

where $\mathrm{Adm}_{\underline{I}}(\mu) \subset \widetilde{W}_{\underline{I}} \backslash \widetilde{W} / \widetilde{W}_{\underline{I}}$ μ-admissible set. //

If $r = 1$, then $\mathcal{M}^{\text{naive}}$ has semi-stable reduction (Drinfeld).

Conjecture: $\exists$ $\mathcal{G}$-equiv. blow up in special fiber of $\mathcal{M}^{\text{naive}}$ w. s-s. reduction (Faltings, $r=2$)

<u>Theorem 2</u>: Let $e \geq 2$. (i) If (r_φ) differ by more than one, then $\mathcal{M}^{naive}$ <u>not flat</u> $/ O_E$.

(ii) If differ by at most one, then $\mathcal{M}^{naive}$ is flat at least in the following cases: $r \leq e$ or $e = 2$ or char $k = 0$ //

To define flat models want closed subschemes of $\mathcal{M}^{naive}$ with same generic fiber. Let

K = Galois closure of F/F_0. Number embeddings $\varphi_1, \dots, \varphi_e : F \to K$.

<u>Splitting model</u> is a project. scheme $\widetilde{\mathcal{M}} / O_K$ which represents the following functor: $S \mapsto$ iso-classes of commut. diagr.

$$
\begin{array}{ccccc}
\Lambda_{i_0,S} & \longrightarrow & \Lambda_{i_1,S} & \longrightarrow & \cdots \\
\cup & & \cup & & \\
\mathcal{F}_0^e & \longrightarrow & \mathcal{F}_1^e & \longrightarrow & \\
\cup & & \cup & & \\
\mathcal{F}_0^{e-1} & \longrightarrow & \mathcal{F}_1^{e-1} & \longrightarrow & \\
\cup & & \cup & & \\
\vdots & & & & \\
\cup & & & & \\
\mathcal{F}_0^0 = (0) & \longrightarrow & & &
\end{array}
$$

Here $\mathcal{F}_i^s$ again loc. direct summands, stable under O_F. Require that $\operatorname{rk} \mathcal{F}_i^s = \sum_1^s r_j$ and

$$\left(\iota(a) - \varphi_s(a)\right)(\mathcal{F}_i^s) \subset \mathcal{F}_i^{s-1}, \quad \forall\forall\, s, i.$$

$G \otimes_{F_0} K = \prod_1^e GL_d$.

Theorem 3: $\widetilde{\mathcal{M}}$ is a twisted product of unramified naive local models relative to GL_d / K, $\mu_i = \omega_{r_i}$,

$$\widetilde{\mathcal{M}}_I = \mathcal{M}_I^1 \tilde{\times}_{O_K} \mathcal{M}_I^2 \tilde{\times}_{O_K} \cdots \tilde{\times}_{O_K} \mathcal{M}_I^e . \quad //$$

[means $\mathcal{M}^1 \times_{O_K} \cdots \times_{O_K} \mathcal{M}^e \longleftarrow \widetilde{\mathcal{M}}' \longrightarrow \widetilde{\mathcal{M}}$ pts under $\mathcal{G}^2 \times_{O_K} \cdots \times_{O_K} \mathcal{G}^e$.

Corollary: $\widetilde{\mathcal{M}}$ is flat $/ O_K$. //

Note that have (keep $(\mathcal{F}_i^e, i \in I)$).

$$\widetilde{\mathcal{M}} \longrightarrow \mathcal{M}^{naive} \otimes_{O_E} O_K \longrightarrow \mathcal{M}^{naive}$$

Let $\mathcal{M}$ = image, is closed subscheme of $\mathcal{M}^{naive}$, with same generic fiber. But no functor description!

Theorem 4: $\mathcal{M}$ is flat $/ O_E$ with reduced special fiber. Each irreducible comp. is normal, CM with rational singularities. Furthermore,

$$\mathcal{M} \otimes_{O_E} k = \bigcup_{w \in Adm_I(\mu)} \mathcal{O}_w$$

Here $\mu = \mu_1 + \cdots + \mu_e$ //

Proof uses Haines / Ngo and Görtz's lifting thm. Let

$$R\Psi_K^{\mathcal{M}} = R\Psi(\mathcal{M} \otimes_{O_E} O_K / O_K) \, \mathbb{Q}_\ell \{d\}$$

$d = \dim \mathcal{M}/O_K$.

Corollary: $R\Psi_K^{\mathcal{M}} = R\Psi_K^{\mathcal{M}^1} * \dots * R\Psi_K^{\mathcal{M}^e}$ //

RHS is known due to Haines / Ngo.

$\exists$ suggestions by Drinfeld / Gaitsgory on descent to O_E.

Remarks: a) The same works for Shimura varieties assoc. to $G = R_{F/\mathbb{Q}}(GSp_{2n})$: uses Faltings's thm on normality of affine Schubert varieties, but also more.

b) For unitary group corresp. to $F/\mathbb{Q}$ ramified in p, serious problems remain.