ON AFFINE DELIGNE-LUSZTIG VARIETIES

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1. Definitions

Let G be a reductive group over $\mathbb{F}_q$. Let B be a Borel subgroup defined over $\mathbb{F}_q$, and let T be a maximal torus contained in B. We denote by a tilde the groups defined over $\overline{\mathbb{F}}_q$ obtained by base change, i.e.,

$$\tilde{G} = G \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \quad , \quad \tilde{B} = B \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \quad , \quad \tilde{T} = T \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \quad . \tag{1.1}$$

Let $W = \mathrm{Norm}(\tilde{T})/\tilde{T}$ be the Weyl group. Then W gives the relative position of two Borel subgroups of $\tilde{G}$, i.e.,

$$\tilde{B}\backslash\tilde{G}/\tilde{B} = W. \tag{1.2}$$

Hence we obtain the map inv as the composition

$$\mathrm{inv} : \tilde{G}/\tilde{B} \times \tilde{G}/\tilde{B} \longrightarrow \tilde{G}\backslash[\tilde{G}/\tilde{B} \times \tilde{G}/\tilde{B}] = \tilde{B}\backslash\tilde{G}/\tilde{B} = W. \tag{1.3}$$

Since $\tilde{B}$ is defined over $\mathbb{F}_q$ we have an action of the Frobenius $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $\tilde{G}/\tilde{B}$.

Definition 1.1. The Deligne-Lusztig variety associated to $w \in W$ is the locally closed subvariety of $\tilde{G}/\tilde{B}$,

$$X_w = \{x \in \tilde{G}/\tilde{B}; \mathrm{inv}(x, \sigma(x)) = w\}.$$

This is a smooth algebraic variety, equidimensional of dimension equal to $\ell(w)$ [2]. The finite group $G(\mathbb{F}_q)$ acts on X_w.

A variant of this construction is obtained as follows. Replace B by a parabolic subgroup P defined over $\mathbb{F}_q$ and containing B. Let $W_P = (\mathrm{Norm}(\tilde{T}) \cap \tilde{P})/\tilde{T}$. Then

$$\tilde{P}\backslash\tilde{G}/\tilde{P} = W_P\backslash W/W_P. \tag{1.4}$$

Correspondingly we define the *generalized Deligne-Lusztig variety associated to P and $\overline{w} \in W_P\backslash W/W_P$* which we denote by $X_{\overline{w},P}$, comp. [3]. This is a smooth locally closed subvariety of $\tilde{G}/\tilde{P}$ of dimension $\ell(\overline{w})$ (length of the Kostant representative of $\overline{w}$).

Now let F be a local field, i.e., a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$. Let L be the completion of the maximal unramified extension of F. We denote by σ the relative Frobenius automorphism in $\mathrm{Gal}(L/F)$. Let G be a reductive group over F. Let $K_0 \subset G(F)$ be an Iwahori subgroup defined over F and denote by $\tilde{K}_0 \subset G(L)$ the

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corresponding Iwahori subgroup over L. Let $\tilde{W}$ be the extended affine Weyl group of $G(L)$, which may be defined as follows.

Let $\tilde{S}$ be a maximal split torus in $\tilde{G}$ such that $\tilde{K}_0$ fixes an alcove in the apartment corresponding to $\tilde{S}$. Let $\tilde{T}$ be the centralizer of $\tilde{S}$, a maximal torus by Steinberg's theorem. Let $\tilde{T}(L)_1$ be the unique Iwahori subgroup of $\tilde{T}(L)$. Then

$$\tilde{W} = \mathrm{Norm}(\tilde{T})(L)/\tilde{T}(L)_1. \tag{1.5}$$

Then

$$\tilde{K}_0\backslash\tilde{G}(L)/\tilde{K}_0 = \tilde{W}. \tag{1.6}$$

Hence we obtain as in (1.3) a map

$$\mathrm{inv} : \tilde{G}(L)/\tilde{K}_0 \times \tilde{G}(L)/\tilde{K}_0 \longrightarrow \tilde{W}. \tag{1.7}$$

The following definition is taken from [13].

Definition 1.2. Let $b \in G(L)$ and $w \in \tilde{W}$. The affine Deligne-Lusztig variety associated to b and w is the set

$$X_w(b\sigma) = \{x \in \tilde{G}(L)/\tilde{K}_0 \ ; \ \mathrm{inv}(x, b\sigma(x)) = w\}.$$

Let

$$J(F) = \{g \in G(L) \ ; \ g^{-1}b\sigma(g) = b\}. \tag{1.8}$$

Then $J(F)$ acts on $X_w(b\sigma)$.

It is clear that if b, b' are σ-conjugate in $G(L)$, then there is a bijection between the corresponding affine DL-varieties $X_w(b\sigma)$ and $X_w(b'\sigma)$. (This, incidentally, explains the absence of b in the original definition of DL-varieties over a finite field: any element $b \in G(\overline{\mathbb{F}}_q)$ is σ-conjugate to the unit element.)

Remark 1.3. So far, $X_w(b\sigma)$ is just a set. But it is expected that $X_w(b\sigma)$ is the set of $\overline{\mathbb{F}}_p$-points of an algebraic variety over $\overline{\mathbb{F}}_p$ which is of finite dimension and locally of finite type. This holds at least in the function field case, compare also the example below. This also holds if $G = GL_n$ or $G = GSp_{2n}$ and the σ-conjugacy class $[b]$ of b lies in $B(G, \mu)$, where μ is minuscule, cf. (2.3) below. In fact, in this case the corresponding affine DL-variety can be identified with the reduced scheme underlying a formal moduli space of p-divisible groups, cf. [15].

A variant of the above definition is obtained as follows. Let $K \subset G(F)$ be a parahoric subgroup defined over F containing K_0 and let $\tilde{K} \subset \tilde{G}(L)$ be the corresponding parahoric subgroup of $\tilde{G}(L)$. Let

$$\tilde{W}_K = (\mathrm{Norm}(\tilde{T})(L) \cap \tilde{K})/\tilde{T}(L)_1. \tag{1.9}$$

Then

$$\tilde{K}\backslash\tilde{G}(L)/\tilde{K} = \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K. \tag{1.10}$$

Hence we obtain as in (1.3) a map

$$\mathrm{inv} : \tilde{G}(L)/\tilde{K} \times \tilde{G}(L)/\tilde{K} \longrightarrow \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K. \tag{1.11}$$

Correspondingly we define *the generalized affine* DL-*variety associated to* K *and to* $\overline{w} \in \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K$, which we denote by $X_{\overline{w}}(b\sigma)_K$.

Example 1.4. Let $G = GL_2$ and $K = GL_2(\mathcal{O}_F)$ and $\tilde{K} = GL_2(\mathcal{O}_L)$. Then

$$\tilde{W}_K \backslash \tilde{W} / \tilde{W}_K = (\mathbb{Z}^2)_+ = \{(\mu_1, \mu_2) \in \mathbb{Z}^2 ; \mu_1 \geq \mu_2\}. \tag{1.12}$$

If $b = 1$, then

$$\begin{aligned} X_{(0,0)}(\sigma)_K &= \{\tilde{\Lambda} \subset L^2 \ ; \ \tilde{\Lambda} \text{ a } \mathcal{O}_L\text{-lattice with } \tilde{\Lambda}^\sigma = \tilde{\Lambda}\} \\ &= \{\Lambda \subset F^2 \ ; \ \text{a } \mathcal{O}_F\text{-lattice}\}. \end{aligned} \tag{1.13}$$

This is the set of vertices in the building of $GL_2(F)$, which is a discrete set.

More generally, let $\mu = (\mu_1, \mu_2) \in (\mathbb{Z}^2)_+$. Then it is easy to see that in the function field case $X_\mu(\sigma)_K$ is empty if $\mu_1 + \mu_2 \neq 0$ and that $\dim X_\mu(\sigma)_K = \frac{1}{2}(\mu_1 - \mu_2)$ if $\mu_1 + \mu_2 = 0$.

The fact that an affine DL-variety can be empty should be contrasted to the finite field case. We are led to the following questions.

1. When is $X_{\overline{w}}(b\sigma)_K \neq \emptyset$?
2. Is $X_{\overline{w}}(b\sigma)_K$ equi-dimensional, and is there a formula for its dimension?
3. What is the global structure of $X_{\overline{w}}(b\sigma)_K$, e.g., what is its singular locus, what are its irreducible components, etc.?

Of course, in questions 2 and 3, we need the structure of an algebraic variety on $X_{\overline{w}}(b\sigma)_K$. Question 1 has been investigated, in part with the help of computer calculations, in cases when K is an Iwahori subgroup by Reuman [16], Görtz, and Lau.

2. Hyperspecial K

To simplify the exposition, we will assume that G is split over F. Denoting by the same letter the canonical integral model of G over $\mathcal{O}_F$, we take $K = G(\mathcal{O}_F)$ and $\tilde{K} = G(\mathcal{O}_L)$.

We first recall some facts from Kottwitz's description of the set $B(G)$ of σ-conjugacy classes in $G(L)$[6][7]. We fix a maximal split torus A and a Borel subgroup B containing A. Let

$$\mathfrak{a} = X_*(A) \otimes \mathbb{Q} \quad , \quad \mathfrak{a}^+ = \mathfrak{a} \cap \overline{C}, \tag{2.1}$$

where $\overline{C}$ is the closure of the positive Weyl chamber relative to the choice of B. In the sequel we shall need two maps

$$\begin{array}{rcccccl} \overline{\nu} & : & B(G) & \longrightarrow & \mathfrak{a}^+ & , & \text{the Newton map} \\ \kappa = \kappa_G & : & B(G) & \longrightarrow & \pi_1(G) & , & \text{the Kottwitz map.} \end{array}$$

Here $\pi_1(G) = X_*(A)/X_*(A_{sc})$ is Borovoi's algebraic fundamental group of G.

For this choice of K we have

$$\tilde{W}_K \backslash \tilde{W} / \tilde{W}_K = X_*(A)_{\mathrm{dom}} \tag{2.2}$$

(the set of dominant cocharacters for this choice of B). Hence to any $\mu \in X_*(A)_{\mathrm{dom}}$ and any $b \in G(L)$ we have associated the corresponding generalized affine DL-variety $X_\mu^G(b\sigma) = X_\mu(b\sigma)_K$. To μ there is associated its image $\mu^\natural \in \pi_1(G)$. Let

$$B(G, \mu) = \{[b] \in B(G) \ ; \ \overline{\nu}_{[b]} \leq \mu \ , \kappa([b]) = \mu^\natural\}. \tag{2.3}$$

Here $\leq$ is the customary partial order on $\mathfrak{a}^+$. Then $B(G, \mu)$ is a finite subset of $B(G)$. It is partially ordered, with a unique minimal element (the unique *basic* element in $B(G, \mu)$) and a unique maximal element (the μ-*ordinary* element).

Proposition 2.1. *If $X_\mu^G(b\sigma) \neq \emptyset$, then the σ-conjugacy class $[b]$ of b lies in $B(G,\mu)$.*

This is the group-theoretic version of Mazur's inequality [14]. Recently Kottwitz [8] gave a purely group theoretic proof based on the positivity lemma of Harish-Chandra and Arthur.

Conjecture 2.2. *The converse holds.*

This is known to be true in the following cases:

(i) G a classical group (Kottwitz, Rapoport [9] for GL_n and Sp_{2n}; Leigh [10] for the orthogonal groups; Fontaine, Rapoport [4] for GL_n)

(ii) G arbitrary, but μ minuscule (Wintenberger [18]).

So much for question 1 in this context. Next we turn to question 2 (dimension). The following formula is inspired by a formula of Chai [1].

Conjecture 2.3. *Let $b \in G(L)$ with $[b] \in B(G,\mu)$. Then $X_\mu^G(b\sigma)$ is equi-dimensional of dimension*

$$\dim X_\mu^G(b\sigma) = \langle 2\rho, \mu - \overline{\nu}_{[b]}\rangle + \sum_{i=1}^{\ell} [\langle \omega_i, \overline{\nu}_{[b]} - \mu\rangle].$$

Here $\omega_1, \dots, \omega_\ell$ are the fundamental weights of the adjoint group G_{ad}.

This is known in a few cases:

(i) Let $b = 1$. Then the formula above predicts

$$\dim X_\mu^G(\sigma) = \langle \rho, \mu\rangle. \tag{2.4}$$

This has been proved by Kottwitz and Reuman, in somewhat greater generality, comp.[17]. In fact, they apparently prove a reduction theorem which shows the validity of the conjecture for (G, b, μ), provided it is known to be true for (M, b, μ), where M is a Levi subgroup of G and $b \in M(L)$ defines an element of $B(M, \mu)$.

(ii) Let $G = GL_n$, μ minuscule. Then the formula has apparently been proved independently by Chai and Oort, cf.[12] and by Mierendorff [11]. Perhaps it would be more prudent to say that Chai and Oort, and Mierendorff have performed (independently, and by completely different methods) a parameter count which supports the conjecture.

There is the following remarkable example. Let $G = GL_n$ and let $\mu = (1, 0, \dots, 0)$. Then for $b \in B(G,\mu)$ basic, i.e., $\overline{\nu}_{[b]} = \frac{1}{n} \cdot (1, \dots, 1)$, we get from the above formula the prediction

$$\dim X_\mu^G(b\sigma) = (n-1) + \sum_{i=1}^{n-1} (-1) = 0. \tag{2.5}$$

It is easy to check directly that this is indeed correct. The question arises as to how often this can happen.

Proposition 2.4. *Let $G = GL_n$. Let $b \in G(L)$ such that $[b] \in B(G,\mu)$ is basic. Then $\dim X_\mu^G(b\sigma) = 0$ if and only if*

$$\mu = \begin{cases} r \cdot \mathbf{1} \\ (1, 0, \dots, 0) + r \cdot \mathbf{1} \\ (1, \dots, 1, 0) + r \cdot \mathbf{1}\ , \quad \textit{some } r \in \mathbb{Z}. \end{cases}$$

Here $\mathbf{1} = (1, \ldots, 1)$. We call the last two alternatives *the Drinfeld case.*

This is an indication as to how rare this is. For instance,

Conjecture 2.5. *Let G be simple, adjoint, and fix μ. Let $b \in G(L)$ such that $[b] \in B(G, \mu)$. Then $\dim X_\mu^G(b\sigma) > 0$ unless either $[b]$ is $\mu-$ordinary or G is PGL_n and μ is of Drinfeld type..*

Even assuming the validity of Conjecture 2.3, this seems difficult to establish.

Remark 2.6. One may expect that $\dim X_\mu^G(b\sigma) = 0$ if and only if $J(F)$ acts transitively on $X_\mu^G(b\sigma)$, cf. Conjecture 2.8 below. As Laumon has pointed out, the transitivity of this action is related to the question in which cases one can calculate in an elementary way the twisted orbital integrals occuring in the fundamental lemma for base change. Taking into account Proposition 2.4 and Conjecture 2.5, one may expect that Drinfeld found the essentially unique cases when such an elementary proof of the fundamental lemma can be expected.

We finally turn to question 3.

Conjecture 2.7. (Mierendorff) *Let $b \in G(L)$ such that $[b] \in B(G, \mu)$. Let M be the unique standard Levi subgroup of G minimal with the property that b is σ-conjugate to an element in $M(L)$ which determines a class in $B(M, \mu)$. Then*

$$\pi_0(X_\mu^G(b\sigma)) = \pi_1(M)^{\langle\sigma\rangle} = \pi_1(M)$$

(the last equality because G and hence M is split).

Mierendorff [11] has proved this when $G = GL_n$ and μ is minuscule.

Conjecture 2.8. (Mierendorff) *The group $J(F)$ acts transitively on the set of irreducible components of $X_\mu(b\sigma)$.*

It seems that Mierendorff [11] will be able to prove this in the case when $G = GL_n$ and μ is minuscule.

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