Korea Nov 2001

On the reduction modulo p of Shimura varieties

Fundamental problem in arithm. algebraic geometry:

X = proj. non-sing. alg. var. / E = nb. field.

Zeta function is Euler product

$$\zeta(X/E, s) = \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(X/E, s) ,$$

in a place $\mathfrak{p}$ of good reduction with model $\mathcal{X}/O_{E_{\mathfrak{p}}}$ have

$$\zeta_{\mathfrak{p}}(X/E, s) = \exp\Big(\sum_{n=1}^{\infty} |\mathcal{X}(\kappa_{\mathfrak{p}^n})| \frac{T^n}{n}\Big) , \quad T = q_{\mathfrak{p}}^{-s} .$$

In the case of bad reduction it is generally believed

Serre: Sem. DPP.

that instead of $|\mathcal{X}(\kappa_{\mathfrak{p}^n})| = \sum_{x \in \mathcal{X}(\kappa_{\mathfrak{p}^n})} 1$ one has to

weight each point x with $\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}}^n ; R\Psi_{\mathcal{X}/O, x}^{I_{\mathfrak{p}}})$.

The conjecture is that the zeta function can be meromorph. continued to the whole s-plane, with functional equation etc.

A little bit of progress has been made for Shimura varieties, because for them

Casselman: Corvallis

Milne; — 4 —

- one can enumerate the points modulo $\mathfrak{p}$
- one has a grip on the singularities in the models

All this at least for Sh.-var. of PEL-type, with parahoric reduction

Principle: Express all findings in grouptheoretic terms.

1. The moduli scheme of elliptic curves

Fix p. For $n \geq 3$ prime to p consider the functor

on $Sch/\mathbb{Z}[\frac{1}{n}]$, $\mathcal{M}$: $S \longmapsto \{$ iso-classes of elliptic curves E/S + $\alpha : E_n \simeq (\mathbb{Z}/n)^2\}$.

Theorem: This functor is representable by a quasi-projective smooth curve over $\mathbb{Z}[\frac{1}{n}]$. One has

$$\mathcal{M}(\mathbb{C}) = \mathfrak{H}_{\pm} / \Gamma(n)$$

To enumerate $\mathcal{M}(\overline{\mathbb{F}}_p)$, we proceed in 2 steps:

a) divide $\mathcal{M}(\overline{\mathbb{F}}_p)$ up into isogeny classes : $(E, \alpha) \sim (E', \alpha') \overset{=}{\text{df}}$

$\exists$ isogeny $E \to E'$.

Classical fact: Isogeny classes are in 1:1-correspondences

with $\begin{cases} \text{the set of imag.-quadr. fields } F \text{ such that } p \text{ splits in } F \\ + \\ 1 \text{ supersingular isogeny class.} \end{cases}$

b) enumerate the elements of $\mathcal{M}(\overline{\mathbb{F}}_p)$ in one isogeny class.

Fix $E/\overline{\mathbb{F}}_p$. Then the set of elements in the isogeny class is described as follows

$$I(\mathbb{Q}) \Big\backslash \left[\prod_{\ell \neq p}{}' \left\{ \begin{array}{c} \text{lattices in } V_\ell(E) \\ \text{with triv. mod } n \end{array} \right\} \times \left\{ \begin{array}{c} (F,V)\text{-lattices in} \\ V_p(E) \end{array} \right\} \right]$$

$$K^p(m) = \{g \in \prod GL_2(\mathbb{Z}_\ell) \; ; \; g \equiv 1 \bmod m\}$$

Here $I(\mathbb{Q}) = \mathrm{End}(E)^\times_{\mathbb{Q}} = \begin{cases} F^\times \\ D^\times \end{cases}$

$V_\ell(E)$ = rational Tate module

Group theoretically, can write first factor

$$\prod_{\ell \neq p}{}' \{\text{lattices}\}_{\text{w. trivializ. mod } m} = GL_2(\mathbb{A}_f^p)/K^p(m)$$

The most interesting ingredient is the last factor. It depends only on p-divisible group X assoc to E. Let $L = \hat{\mathbb{Q}}_p^{un}$, O_L with Frobenius action σ.

Demazure: SLN. To X can assoc. its Dieudonné module M which is a free O_L-module of rank 2 with σ-linear operator $F: M \to M$ such that $pM \subset FM \subset M \Leftrightarrow FM \subset M, \; VM \subset M$ where $V = p \cdot F^{-1}$.

By $V_p(E)$ we have denoted the rational $\mathcal{D}$-module $M \otimes_{O_L} L$: this a L-vector space of dim. 2 with bijective σ-linear F.

Group-theoretically the second factor can be written as follows.

Consider $G(L)/G(O) = \{\text{set of lattices } \Lambda \subset L^2\}$,

inv: $G(L) \backslash [G(L)/G(O) \times G(L)/G(O)] = G(O) \backslash G(L)/G(O) = \{(j,k) \in \mathbb{Z}^2; \; j \geq k\}$
$= X_*(T)/W$

(elementary divisor theorem)

The 2^{nd} factor is

$$\{ g \in G(L)/G(\mathcal{O}) \;;\; \mathrm{inv}(g, Fg) = \overset{\mu}{\overset{\shortparallel}{(1,0)}} \}.$$

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From here on to $\zeta_p(\mathcal{M}/\mathbb{Z}[\frac{1}{m}], s)$ is purely combinatorial/group theoretical — not my business.

<u>Variant</u> $\Gamma_0(p)$: Consider the functor $\mathcal{M}'$ over $\operatorname{Spec} \mathbb{Z}[\frac{1}{m}]$,

$S \mapsto \{$ iso-classes of isogenies of degree p $E_0 \to E_1$, $\alpha : E_{i,m} \cong (\mathbb{Z}/m)^2 \}$

<u>Theorem</u>: This functor is representable by a quasi-projective curve over $\mathbb{Z}[\frac{1}{m}]$, with semistable reduction at p. One has

much more
Deligne–Rapo: SLN

$$\mathcal{M}'(\mathbb{C}) = \mathfrak{h}_{\pm} / (\Gamma(m) \cap \Gamma_0(p)).$$

The description of $\mathcal{M}'(\overline{\mathbb{F}}_p)$ is along the same lines as for $\mathcal{M}(\overline{\mathbb{F}}_p)$. Namely a) is identical and in b) only the 2^{nd} factor changes. Namely for Dieudonné modules have

$$(*) \quad \begin{array}{ccccc} pM_0 & \subset & FM_0 & \subset & M_0 \\ \cup & & \cup & & \cup \\ pM_1 & \subset & FM_1 & \subset & M_1 \end{array}$$

To describe this, consider

$$G(L)/K_0 = \{ \text{lattices } \Lambda_0, \Lambda_1 \text{ in } L^2 \text{ such that } p\Lambda \subsetneq \Lambda_1 \subsetneq \Lambda_0 \}.$$

$$\uparrow$$

$$\mathrm{pr}^{-1}\left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right) \subset G(\mathcal{O}).$$

Here is a sophisticated way of checking the smoothness of $\mathcal{M}$. Need to check infinitesimal criterion. Let $S_0 \subset S$ nilpotent thickening with $I^2 = (0)$. Let E/S_0 elliptic curve. Consider Hodge filtration of loc. free $\mathcal{O}_{S_0}$-modules of rank 1, 2 resp,

$$\begin{array}{ccc} \mathrm{Fil}^1 & \subset & H_1^{DR}(E/S_0) \\ \| & & \| \\ H^1(E_0, \mathcal{O})^* & \to & H^1(\Omega^\bullet_{E/S_0}) \end{array}$$

Messing: SLN for statement

By Groth./Messing have canonical lifting of $H_1^{DR}(E/S_0)$ to $\mathcal{O}_S$-module loc. free of rank 2. Then

$$\{\text{liftings of } E \text{ to } S\} \xrightarrow{\sim} \{\text{liftings of } \mathrm{Fil}^1\}.$$

It follows that deformation theory of E/S_0 is the same as $\mathbb{P}(H_1^{DR}(E/S_0)) \approx \mathbb{P}^1$. In particular, smooth.

Then

$$\mathrm{inv} : G(L) \backslash \left[G(L)/K_0 \times G(L)/K_0 \right] = K_0 \backslash G(L)/K_0 \xrightarrow{\sim} \tilde{W} = X_*(T) \rtimes W$$

Here $\tilde{W}$ is the extended affine Weyl group

$$\tilde{W} = W_a \rtimes \mathbb{Z}.$$

The elements of W_a can be described by their action on the apartment of T (Coxeter group)

In our case at hand, $\mathrm{inv}\,((M_0, M_1), F(M_0, M_1))$ is no longer a single element, but has 3 elements

$$\mathrm{Adm}(\mu) = \{ t_{(1,0)},\ t_{(0,1)},\ s \cdot t_{(1,0)} \}$$
$$= \{ w \in \tilde{W};\ w \leq t_\mu \text{ or } w \leq t_{s(\mu)} \}.$$

Hence 2^{nd} factor is group theoretically

$$\{ g \in G(L)/K_0 \ ;\ \mathrm{inv}(g, Fg) \in \mathrm{Adm}(\mu) \}.$$

~~Finally the singularities of M' are the same as those~~

~~of~~

$$\cancel{M'^{loc} = \mathbb{P}(M_1) \vee \mathbb{P}(M_2)}.$$

$$\cancel{= \text{closure of } \mathbb{P}^1 \text{ in } \mathbb{P}(M_1) \times_{\mathrm{Spec}\, \mathcal{O}} \mathbb{P}(M_2)}$$

~~(local model)~~

To understand the singularities of $\mathcal{M}'$, we introduce the corresponding local model. Let

$$\Lambda_0 = \mathrm{span}\{e_1, e_2\}, \quad \Lambda_1 = \mathrm{span}\{p^{-1}e_1, e_2\} \quad \text{lattices in } \mathbb{Q}_p^2.$$

Hence have lattice chain

$$\Lambda_0 \subset \Lambda_1 \subset \tfrac{1}{p}\Lambda_0$$

Let

$$\mathcal{M}'^{loc}(S) = \left\{ \begin{array}{ccccc} \Lambda_0 \otimes \mathcal{O}_S & \to & \Lambda_1 \otimes \mathcal{O}_S & \xrightarrow{p} & \Lambda_0 \otimes \mathcal{O}_S \\ \cup & & \cup & & \cup \\ \mathcal{F}_0 & \xrightarrow{u_1} & \mathcal{F}_1 & \xrightarrow{u_0} & \mathcal{F}_0 \end{array} \text{ commut.} \quad \text{s.t. } \mathcal{F}_i \in \mathbb{P}(\Lambda_i)(S) \right\}.$$

Then $\mathcal{M}'^{loc} \subset \mathbb{P}(\Lambda_0) \times_{\mathbb{Z}_p} \mathbb{P}(\Lambda_1)$, with $\mathcal{M}'^{loc}_{\mathbb{Q}_p} = \mathbb{P}^1_{\mathbb{Q}_p}$.

Let

$$\widetilde{\mathcal{M}}'(S) = \left\{ (E_0 \to E_1, \alpha) \in \mathcal{M}'(S), \ + \text{ isomorph} \quad \begin{array}{ccccc} \Lambda_0 \otimes \mathcal{O}_S & \to & \Lambda_1 \otimes \mathcal{O}_S & \to & \Lambda_0 \otimes \mathcal{O}_S \\ \wr\| & & \wr\| & & \wr\| \\ H_1^{DR}(E_0/S) & \to & H_1^{DR}(E_1/S) & \to & H_1^{DR}(E_0/S) \end{array} \right\}.$$

Then get diagram

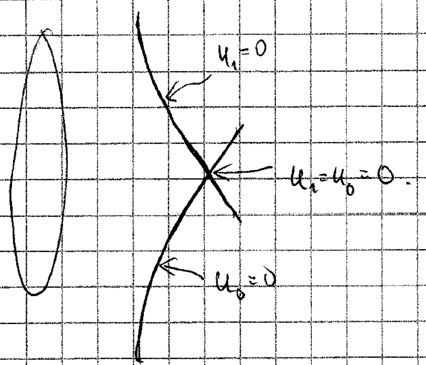
RZ, ch 3

$$\begin{array}{ccc} & \widetilde{\mathcal{M}}' & \\ \mathcal{G} = \underline{\mathrm{Aut}}(\Lambda_\bullet) & \pi \downarrow & \searrow^{\varphi} \\ & \mathcal{M}' & \qquad \mathcal{M}'^{loc} \end{array}$$

Then: φ is smooth of relative dimension $\dim \mathcal{G}$. In particular, $\forall x \in \mathcal{M}' \ \exists y \in \mathcal{M}'^{loc}$ and isomorphic étale nbhds

$$\mathcal{U}'_x \simeq \mathcal{U}'^{loc}_y.$$

The singularities of $\mathcal{M}'^{loc}$ are easy to analyze.



From now on wants to concentrate on p-aspects of this problem. In particular, we only keep

G = reductive algebraic group / $\mathbb{Q}_p$, e.g. GL_2.

μ = conj.-class of cocharact. of G, e.g. $\mu = (1,0)$

K = parahoric subgp. def. over $\mathbb{Q}_p$, e.g. $G(\mathbb{Z}_p)$ or K_0.

Let E = field of definition of μ.

§2. Local models.

Aim: Start with (G, μ, K). To K is assoc. smooth group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ s.t. $\mathcal{G}(\mathbb{Z}_p) = K$. Want, at least for μ minuscule, a projective scheme $\mathcal{M}^{loc} = \mathcal{M}^{loc}(G,\mu)_K$ over O_E w. action of $\mathcal{G}_{O_E}$, s.t.

(i) $\mathcal{M}^{loc}$ is flat w. generic fiber $\mathcal{M}^{loc} \otimes_{O_E} E = G/P_\mu$.

(ii) $\mathcal{M}^{loc}(\bar{k}) = \{ g \in G(L)/\tilde{K} \ ; \ \tilde{K} g \tilde{K} \in \mathrm{Adm}_{\tilde{K}}(\mu) \}$.

(iii) functorial in K and in G.

(iv) $\mathcal{M}^{loc}(G,\mu)_K$ has the same singularities as any Shimura variety giving rise to (G,μ,K).

Typical example:

$G = GL(V)$, $\dim_{\mathbb{Q}_p} V = n$. Let $e_1, \dots, e_n$ basis of V.

Let

$$\Lambda_i = \mathrm{span}_{\mathbb{Z}_p}\{p^{-1}e_1, \dots, p^{-1}e_i, e_{i+1}, \dots, e_n\} \qquad i = 0, \dots, n.$$

Get

$$\Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda_{n-1} \subset \Lambda_n = \tfrac{1}{p}\Lambda_0$$

Minuscale $\mu \triangleq (1^r, 0^{n-r})$ $\quad 0 \le r \le n-1$, $\quad E = \mathbb{Q}_p$.

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Up to conjug., K corresponds to subset

$I = \{0 \le i_0 < i_1 < \dots < i_{m-1} \le n-1\}$ of $\{0, \dots, n-1\}$.

Then $K_I = \{g \in G(\mathbb{Q}_p);\ g\Lambda_i = \Lambda_i,\ \forall i \in I\}$

and $\mathcal{G} = \mathcal{G}_I = \underline{\mathrm{Aut}}(\Lambda_{i_j})$.

<u>The local model is now defined by following</u>

<u>moduli problem on</u> $(Sch/\mathbb{Z}_p)$: $S \mapsto$

$$\left\{ \begin{tikzcd} \Lambda_{i_0}\otimes O_S \arrow[r] & \Lambda_{i_1}\otimes O_S \arrow[r] & \cdots \arrow[r] & \Lambda_{i_{m-1}}\otimes O_S \arrow[r] & \Lambda_{i_0}\otimes O_S \\ \mathcal{F}_{i_0} \arrow[u, phantom, "\cup"] \arrow[r] & \mathcal{F}_{i_1} \arrow[u, phantom, "\cup"] \arrow[r] & \cdots \arrow[r] & \mathcal{F}_{i_{m-1}} \arrow[u, phantom, "\cup"] \arrow[r] & \mathcal{F}_{i_0} \arrow[u, phantom, "\cup"] \end{tikzcd} \right\}$$

s.t. $\mathcal{F}_{i_j}$ loc. direct sum. of $\Lambda_{i_j}\otimes O_S$, of rank r.

<u>Obvious</u>: M^{loc} is repres. by a projective scheme over $\mathbb{Z}_p$,

$M^{loc} \subset \prod_{j=0}^{m-1} Grass_r(\Lambda_{i_j})$, and $M^{loc}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = Grass_r(\mathbb{Q}_p^n)$.

Obviously for $I \subset I'$ have $M^{loc}_{I'} \longrightarrow M^{loc}_{I}$.

If $I = \{0\}$, then $M^{loc} = Grass_r(\Lambda_0)$ smooth.

Most interesting case $I = \{0, \dots, n-1\}$: then K_I Iwahori subgr.

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<u>Theorem (Görtz)</u>: $\mathcal{M}^{loc}$ is flat over Spec $\mathbb{Z}_p$, with reduced special fiber. The irreducible components of the special fiber are normal w. rational singularities, in partic. CM.

<u>Conjecture</u>: $\mathcal{M}^{loc}$ is CM.

Is true for $n \leq 4$. Can be reduced to the following question: Let $B_0, \dots, B_{n-1}$ generic $r \times r$-matrices.

Let

Faltings Crelle

$$M' = \operatorname{Spec} \mathbb{Z}_p[B_0, \dots, B_{n-1}] / B_{n-1} \cdots B_0 = B_{n-2} \cdots B_0 B_1 = \dots = p.$$

Then if $\dim M' \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \dim M' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = (n-1) \cdot (r)^2$ and if $M' \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is CM, then also $\mathcal{M}^{loc}$ is CM.

<u>Special case</u>: $r = 1$. In this case $\mathcal{M}^{loc}$ is semi-stable (Drinfeld case).

<u>This explains point (i)</u>

<u>Ad (iv)</u>: Let $\underline{G}$ = gp of unitary similitudes w.r.t $E/\mathbb{Q}$ of index $(r, n-r)$.

Assume $p = \mathfrak{p} \cdot \mathfrak{p}'$ splits in E.

Then have a model of $Sh(\underline{G},\{h\})_{\underline{K}}$, where $\underline{K} = K^p . K_p$ where $K_p = K$ is parahoric, over $\mathbb{Z}_p = \mathcal{O}_{E_p}$ and (iv) holds.

In partic., Drinfeld case corresp. to signature $(1,n-1)$.

<u>To explain point (ii)</u>, partially at this point, consider the special fiber $\mathcal{M}^{loc} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$, <u>in the Iwahori case to simplify</u>.

~~$\tilde{\Lambda}_i = \mathrm{span}_{\mathbb{F}_p[[t]]} \{t^{-1}e_1, \dots, t^{-1}e_i, e_{i+1}, \dots, e_n\}$.~~

~~Then $\tilde{\Lambda}_0 \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = \Lambda_0 \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Hence to~~

~~$\mathcal{F}_0 \in \mathcal{M}'(S)$ with $S = \mathrm{Spec}\, R$ where R $\mathbb{F}_p$-algebra~~

~~$\tilde{\mathcal{F}}_0 = pr^{-1}(\mathcal{F}_0) \subset \tilde{\Lambda}_i \otimes_{\mathbb{F}_p} R$~~

Recall the affine flag variety $\mathcal{F}$ over Spec $\mathbb{F}_p$,

$$\mathcal{F}(R) = \{\text{complete periodic flags } \mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots \subset \mathcal{L}_{n-1} \subset \mathcal{L}_0\}$$

Here $\mathcal{L}_i$ are free $R[[t]]$-submodules of rank of $R((t))^n$, and

$\mathcal{L}$.

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Consider LG as ind-group scheme over $\mathbb{F}_p$, with

$$LG(R) = G(R((t))).$$

Have L^+G as subgroup scheme and

$$\mathcal{B} = pr^{-1}(\text{Borel in } G \otimes \mathbb{F}_p).$$

Let $\mathcal{F} = LG/\mathcal{B}$ affine flag variety. Then $\mathcal{F}$ represents the following functor. First recall:

- a lattice in $R((t))^n \overset{=}{\underset{\text{Df.}}{}}$ sub-$R[[t]]$-module $\mathcal{L}$ of $R((t))^n$ which is proj. of rank n s.t. $\mathcal{L} \otimes_{R[[t]]} R((t)) = R((t))^n$.

  $\iff$ sub-$R[[t]]$-module $\mathcal{L}$ of $R((t))^n$ s.t. $t^N \cdot R[[t]]^n \subset \mathcal{L} \subset t^{-N} \cdot R[[t]]^n$, some N and s.t. $t^{-N} R[[t]]^n / \mathcal{L}$ loc. free R-module.

- a complete lattice chain in $R((t))^n$ is a sequence of lattices

$$\mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots \subset \mathcal{L}_{n-1} \subset t^{-1} \cdot \mathcal{L}_0$$

s.t. $\mathcal{L}_i / \mathcal{L}_{i-1}$ loc. free R-module of rank 1.

Görtz, B-L: Comm. Math. Ph.

Then $\mathcal{F}(R) = \{ \text{complete lattice chains in } R((t))^n \}$.

Return to $M^{loc} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Then identifying $\Lambda_0 \otimes_{\mathbb{Z}_p} R$ with $R[[t]]^n / t \cdot R[[t]]^n$, let $\mathcal{L}_i$ = inverse image of

$\mathcal{F}_i$. In this way get

$$\iota : M^{loc} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \hookrightarrow \mathcal{F}.$$

Furthermore $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \mathcal{B}$. Hence get as image of ι a finite-dimensional $\mathcal{B}$-invariant closed subset.

Let

$$\tilde{W} = \mathbb{Z}^n \rtimes S_n$$

extended affine Weyl group of GL_n. Then

$$\mathcal{F} = \dot{\bigcup_{w \in \tilde{W}}} X_w$$

$\mathcal{B}$-orbit decomposition. And $\overline{X}_w = \bigcup_{w' \leq w} X_{w'}$.

Hence may write

$$M^{loc} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \bigcup_{w \in Adm(\mu)} X_w ,$$

where $Adm(\mu)$ is a certain finite subset of $\tilde{W}$ closed under $\leq$.

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Proposition (Kottwitz, R): Have

$$Adm(\mu) = \{ w \in \tilde{W} ; \ w \leq t_{\mu'} , \ \mu' = \sigma(\mu), \text{some } \sigma \in S_n \}.$$

$G = GSp_{2r}$, $\mu = (1^r, 0^r)$. Paraho<u>ric</u>:

Görtz: Math. Archive

Analogue for symplectic group ✓ Let now $n = 2r$ and fix

symplectic form s.t. $\langle e_i, e_j \rangle = \begin{pmatrix} 0 & \begin{smallmatrix} & & 1 \\ & 1 & \\ -1 & & \end{smallmatrix} \\ \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & \end{pmatrix}$

Then the lattice chain is selfdual, i.e.

$$\Lambda_0^* = \Lambda_0 \,, \quad \Lambda_r^* = p\Lambda_r \,, \quad \Lambda_i^* = p\,\Lambda_{n-i} \,, \; i = 1, \dots, r-1.$$

Now the local model is defined as before for this r,

with the additional duality condition: For $0 \le i \le r$, $i \in I$

$$\mathcal{F}_i \to \Lambda_i \otimes \mathcal{O}_S \simeq \Lambda_{2r-i}^* \otimes \mathcal{O}_S \to \mathcal{F}_{2r-i}^* \quad \text{is zero.}$$

interpret.

Now fix $I = \{0 \le i_0 < i_1 < \dots < i_{m-1} \le n-1\}$, s.t. for any $i \in I$

with $1 \le i \le r-1$, also $2r - i \in I$. Hence get $K_I \subset G(\mathbb{Q}_p)$.

Theorem (Görtz): Same statement as for GL_n-local model.
For $I = \{0\}$, or $I = \{r\}$ smooth, oth. not.

In this case $\mathcal{M}^{loc} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \hookrightarrow \mathcal{F}(GSp_{2r})$ affine flag variety

And again union of Schubert cells over $\mathrm{Adm}(\mu)$.

This is the local model for $\underline{G} = GSp_{2r}$, $\{\mu\}$. If

$I = \{0\}$, then $\mathcal{M}$ = moduli space of p.p. abelian var. of dim r
+ level-str. prime to p.

If $I = \{0, \dots, 2r-1\}$, the moduli problem is

$$\begin{array}{ccccccc} A_0 & \xrightarrow[\deg p]{} & A_1 & \to & \dots & \to & A_r \\ \downarrow \simeq & & & & & & \downarrow \simeq \\ \hat{A} & \leftarrow & \hat{A}_1 & \dots & \dots & \leftarrow & \hat{A}_r \end{array}$$

full circle $= p$.

Conjecture: In either case $\exists$ $\mathcal{G}$-equivariant blow up of M^{loc} which is semistable $/\mathbb{Z}_p$.

True for $(GL_4, \mu = (1,1,0,0))$, $(GSp_6, \quad)$

Faltings ? Genestier

Complements to previous talk.

a) <u>The embedding</u> $\iota : M^{loc}(GL_n, \mu)_{K_0} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \hookrightarrow \mathcal{F}(GL_n)$

Here $\mu = (1^r, 0^{n-r})$ and K_0 = Iwahori.

Let R $\mathbb{F}_p$-algebra. Have chain of standard lattices in $R((t))^n$,

$$\lambda_{0R} \subset \lambda_{1R} \subset \dots \subset \lambda_{n-1\,R} \subset \lambda_{nR} = t^{-1}\lambda_{0R}$$

where $\lambda_{iR} = \mathrm{span}_{R[[t]]}\{ t^{-1}e_1, \dots, t^{-1}e_i, e_{i+1}, \dots, e_n \}$.

After setting $t = 0$, obtain

$$\begin{array}{ccccccc}
\bar\lambda_{0R} & \to & \bar\lambda_{1R} & \to \cdots \to & \bar\lambda_{n-1\,R} & \to & \bar\lambda_{0R} \\
\| & & \| & & \| & & \| \\
\Lambda_0 \otimes_{\mathbb{Z}_p} R & \to & \Lambda_1 \otimes_{\mathbb{Z}_p} R & \to \cdots \to & \Lambda_{n-1} \otimes_{\mathbb{Z}_p} R & \to & \Lambda_0 \otimes_{\mathbb{Z}_p} R \\
\cup & & \cup & & \cup & & \cup \\
\mathcal{F}_0 & \to & \mathcal{F}_1 & \cdots & \to \mathcal{F}_{n-1} & \to & \mathcal{F}_0
\end{array}$$

If $\mathcal{F}_\bullet \in M^{loc}(R)$ get $\mathcal{L}_i$ = inverse image in λ_{iR} via identific.

Then $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots \subset \mathcal{L}_{n-1} \subset t^{-1}\mathcal{L}_0$ is a complete lattice chain, i.e. an element of $\mathcal{F}(R)$. In this way we obtain an identification

$$M^{loc}(R) = \{ \mathcal{L}_\bullet \in \mathcal{F}(R) \,;\, t\cdot\lambda_i \subset \mathcal{L}_i \subset \lambda_i \,,\ \forall i \in \{0,\dots,n-1\} \}$$

<u>Equivariance</u>: $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \mathcal{B} \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p$ and the action of $\mathcal{B}$ on $\mathrm{Im}(\iota)$ factors through $\mathcal{B} \otimes \mathbb{F}_p$ and is compat. with $\mathcal{G} \otimes \mathbb{F}_p$-action.

Something entirely similar can be done for partial flag varieties, so for any $I \subset \{0, .., n-1\}$ get

$$\mathcal{M}_I^{loc} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \hookrightarrow \mathcal{F}_I = LG / P_I .$$

In particular, for $I = \{0\}$ get embedding into affine Grassmannian.

b) Explanation of Görtz's proof, 2nd attempt: Want to prove that $\mathcal{M}^{loc}$ is reduced (Iwahori case). Consider

$$p_I : \mathcal{F} \longrightarrow \mathcal{F}_I \qquad \text{smooth projective.}$$

Let $\tilde{\mathcal{M}}_I = p_I^{-1}(\mathcal{M}_I \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$. Then obviously

$$\tilde{\mathcal{M}}_I(R) = \{ \mathcal{L}_\bullet \text{ complete} ; \ t \cdot \lambda_{i_R} \subset \mathcal{L}_i \subset \lambda_{i_R} , \ \forall i \in I \}$$

Hence

$$\mathcal{M}^{loc} = \bigcap_{i=0}^{n-1} \tilde{\mathcal{M}}_{\{i\}} \quad \text{and also} \quad \mathcal{M}^{loc} = \tilde{\mathcal{M}}_{\{0\}} \cap \bigcap_{i=1}^{n-1} \tilde{\mathcal{M}}_{\{0,i\}}$$

Argue over first intersection: $\mathcal{M}_{\{i\}}$ is smooth, hence reduced and a Schubert variety. Hence $\tilde{\mathcal{M}}_{\{i\}}$ is reduced and a union of Schubert cells (scheme-theoretically).

By Faltings any (large enough) Schubert variety is normal (hence) and VF-split, compatibly with all smaller Sch.-varieties. And unions and intersections of compatibly F-split subvarieties are again. Hence $\mathcal{M}^{loc}$ is F-split and hence reduced and scheme-th. a union of Sch-varieties

One can avoid Faltings by using the 2^{nd} way of presenting $\mathcal{M}^{loc}$: now $\tilde{\mathcal{M}}_{\{0\}}$ is normal and one has to prove directly that $\mathcal{M}_{\{0,i\}}$ are normal (Mehta / Trivedi, Strickland).

c) <u>Conjecture</u> : <u>Let G be split (more generally quasi-split and split over an unramified extension of $\mathbb{Q}_p$). Then $\exists$ $\mathcal{G}$-equivariant blow-up in special fiber</u>

$$\mathcal{M}^{loc}(G,\mu)_K^{\sim} \longrightarrow \mathcal{M}^{loc}(G,\mu)_K$$

<u>which is semistable over</u> Spec O_E.

Obvious for $(GL_n, (1,0,\ldots,0), K)$ (Drinfeld case), proved for

$(GL_4, (1,1,0,0), K)$ Görtz + Faltings ?
$(GL_5, (1,1,0,0,0), K)$ "
GSp_4 de Jong: JAG
GSp_6 Genestier: Compos.

d) <u>More explanation on motivation.</u>

Let E local field, w. finite res.-field $k = \mathbb{F}_q$. Set

$$1 \to I \to \Gamma \to Gal(\bar{k}/k) \to 1.$$

Let (V,ρ) = f.-d. continuous ℓ-adic rep'n of Γ. A filtr. $\mathcal{F}$ on V is called <u>admissible</u>, if $\mathcal{F}$ Γ-stable and I acts on $gr^{\mathcal{F}}_\cdot(V)$ through a finite quotient.

Grothendieck $\Rightarrow$ a subgr. of finite index $I' \subset I$ acts unipotently, i.e.

$$\rho(g) = \exp(t_\ell(g) N) \quad , \quad g \in I'.$$

where $t_\ell : I \longrightarrow \mathbb{Z}_\ell(1)$ canon. homom. Hence admissible filtr. always exist. Let

$$Tr^{ss}(Fr_q ; V) \underset{\text{Df.}}{=} \sum_k Tr(Fr_q ; gr_k^{\mathcal{F}}(V)^I) \quad \text{- indep't of filtration } \mathcal{F}.$$

This is additive on exact sequences!

Now let X / O_E be a proper scheme w. smooth generic fibre. Then

$$H^i(X_{\bar{E}}, \mathbb{Q}_\ell) = H^i(X_{\bar{k}}, R\Psi).$$

To calculate local factor of Hasse-Weil zeta function, have to calculate

$$\sum_i (-1)^i \, Tr(Fr_q ; H^i(X_{\bar{E}}, \mathbb{Q}_\ell)^I).$$

Very hard!, because no relation to special fiber. Much easier is

$$\sum_i (-1)^i \, Tr^{ss}(Fr_q ; H^i(X_{\bar{E}}, \mathbb{Q}_\ell)) = \sum_i (-1)^i \, Tr^{ss}(Fr_q ; H^i(X_{\bar{k}}, R\Psi))$$

$$= \sum_{x \in X(k)} Tr^{ss}(Fr_q ; R\Psi_x).$$

<u>Fact</u>: WMC for X allows one to recover Tr from Tr^{ss}

<u>Example</u>: Let X = elliptic curve w. split multiplic. reduction. Then $H^1(X_{\bar{E}}, \mathbb{Q}_\ell)$ has 2-step filtration which is admissible,

$$1 \to \mathbb{Q}_\ell \to H^1 \to \mathbb{Q}_\ell(1) \to 0$$

Hence

$$\mathrm{Tr}\,(\mathrm{Fr}_q, H^1(X_{\bar{E}}, \mathbb{Q}_\ell)^I) = 1$$

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, H^1(X_{\bar{E}}, \mathbb{Q}_\ell)) = 1+q.$$

Recover 1^{st} from 2^{nd} by only keeping the contrib. of wt 0.

In the same vein one can define the semi-simple L-function resp. zeta function, with similar relation to true zeta function.

e) Let $\mathcal{M}^{loc}(G,\mu)_{K_0} / O_E$. From the above want to know
Iwahori

for $x \in \mathcal{M}(\mathbb{F}_q)$ the value of :

$$x \longmapsto \mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi_x).$$

Recall stratification (for $G = GL_n$ or $G = GSp_{2n}$),

$$\mathcal{M}^{loc}(\mathbb{F}_q) = \bigcup_{w \in \mathrm{Adm}(\mu)} X_w(\mathbb{F}_q) \subset \mathcal{F}(G)(k) .$$

The above function only depends on $\mathcal{B}$-orbit, hence get function

$$f_\mu : \tilde{W} \longrightarrow \mathbb{Q}_\ell ,$$

with $\mathrm{supp}(f_\mu) \subset \mathrm{Adm}(\mu)$.

Recall the Iwahori-Hecke algebra of G (G split over $\mathbb{F}_q$):

$\mathcal{H}(G(\mathbb{F}_q((t)))//\mathcal{B}(\mathbb{F}_q))$ = space of $\mathcal{B}(\mathbb{F}_q)$-biinvariant functions with compact support

with product = convolution, via $\mathrm{vol}(\mathcal{B}(\mathbb{F}_q)) = 1$.

Now $\mathcal{B}(\mathbb{F}_q)\backslash G(\mathbb{F}_q((t)))/\mathcal{B}(\mathbb{F}_q) = \tilde{W}$ extended affine Weyl grp, hence $\mathcal{H}$ has $\mathbb{Q}_\ell$-basis $\{T_w, w \in \tilde{W}\}$. Hence can interpret f_μ as an element of $\mathcal{H}$.

Recall exact sequence

$$1 \to X_*(T) \to \tilde{W} \to W \to 1$$

<u>Bernstein</u>: $\quad Z(\mathcal{H}) = \mathbb{C}[X_*(T)]^W$ polynomial algebra

$\left(= \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}\right.$ symmet.

math. Arch.

"Kottwitz conjecture"

<u>Theorem (Haines / Ngô)</u>: <u>Let</u> $G = GL_n$ or $G = GSp_{2r}$. <u>Then</u> f_μ <u>lies in the center of</u> $\mathcal{H}$ <u>and</u>

$$f_\mu = q^{\dim M^{loc}} \cdot z_\mu .$$

<u>Here</u> $\mu \mapsto \sum_{w \in W/W_\mu} w\mu \;\hat{=}\; z_\mu \in Z(\mathcal{H})$

The proof relies on extending local models to non-minuscule weights (relation to Gaitsgory: Invent. math.).

~~Let $G = GL_n$, $\mu = (1, 0, \dots, 0)$, $K = K_0$. Then Haines has explicit formula for z_μ as follows.~~

$\boxed{\text{Pictures}}$: transparencies.

3. Local models in the ramified case

We only consider the case where $G = \mathrm{Res}_{F/\mathbb{Q}_p}(GL_n)$, μ minuscule, where $F/\mathbb{Q}_p$ tot. ramif. (the case $G = \mathrm{Res}_{F/\mathbb{Q}_p} GSp_{2r}$, μ minuscule is similar).

Let $F/\mathbb{Q}_p$ tot. ramified of degree e, let $\pi \in O_F$ uniform., root of Eisenstein polynomial,

$$Q(T) = T^e + \sum_{k=0}^{e-1} b_k T^k \qquad , \; b_0 \in \pi_0 \mathbb{Z}_p^\times, \; b_i \in (p)$$

Let V be F-VS of di n; $e_1, \dots, e_n$ basis, get O_F-lattice chain

$$\Lambda_0 \subset \dots \subset \Lambda_{n-1} \subset \pi^{-1}\Lambda_0$$

and for $\emptyset \neq I \subset \{0, \dots, n-1\}$ a partial chain $\Lambda_I \rightsquigarrow K_I \subset GL(V)(F) = R_{F/\mathbb{Q}_p} GL(V)(\mathbb{Q}_p)$

parahoric subgp

To give μ, have $\forall$ embedding $\varphi : F \longrightarrow \overline{\mathbb{Q}}_p$ an integer

$$0 \leq r_\varphi \leq d$$

$\rightsquigarrow E/\mathbb{Q}_p$ s.t. $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E) = \{\tau \in \mathrm{Gal} \; ; \; r_{\tau\varphi} = r_\varphi, \; \forall \varphi\}$

Can define naive local model much as before: $S \in (\mathrm{Sch}/O_E) \rightsquigarrow$

$$\begin{array}{ccccccc}
\Lambda_{i_0} \otimes O_S & \rightarrow & \dots & \rightarrow \Lambda_{i_{m-1}} \otimes O_S & \xrightarrow{\pi} & \Lambda_{i_0} \otimes O_S \\
\cup & & & \cup & & \cup \\
\mathcal{F}_0 & \rightarrow & \dots & \rightarrow \mathcal{F}_{m-1} & \rightarrow & \mathcal{F}_0
\end{array}$$

Here $\mathcal{F}_i$ are $O_F \otimes O_S$-submodules, s.t.

- loc. direct summands as $\mathcal{O}_S$-modules of rank $r = \sum_\varphi r_\varphi$
- $\mathrm{char}(\pi \mid \mathcal{F}_i) = \prod_\varphi (T - \varphi(\pi))^{r_\varphi} \in \mathcal{O}_E[T] \to \mathcal{O}_S[T]$

$\forall i = 0, \dots, n-1.$

Hence get $M_I^{loc} = M^{naive}(G,\mu)_K$ over $\mathrm{Spec}\, \mathcal{O}_E$, projective w. generic fibre = F-inv. subspaces of V which under

$$V \otimes \overline{\mathbb{Q}_p} = \bigoplus_\varphi V_\varphi \quad \text{have rank } (r_{\varphi_1}, \dots, r_{\varphi_e}).$$

Has action of $\mathcal{G}_I \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ on it.

<u>Conjecture:</u> <u>If all r_φ differ amongst each other by at most 1, then $M^{naive}(G,\mu)_K$ is flat $/\mathcal{O}_E$.</u>

On the other hand, if this condition is <u>not</u> satisfied, then M^{naive} is <u>not</u> flat $/\mathcal{O}_E$. <u>Reason:</u> $\dim M^{loc} \otimes_{\mathcal{O}_E} k > \dim M^{loc} \otimes_{\mathcal{O}_E} E$.

Pappas. J AlG

This is done by constructing an closed immersion

$$i : M_I^{loc} \otimes_{\mathcal{O}_E} k \hookrightarrow \mathcal{F}_I ,$$

in the same way as before by identifying

$$\begin{tikzcd}
\Lambda_i \otimes_{\mathbb{Z}_p} R = \lambda_{iR} \otimes_{R[[t]]} R[[t]]/(t^e) & \lambda_{iR} \arrow[l] \\
\cup & \cup \\
\mathcal{F}_i & \mathcal{L}_i \arrow[l] \\
 & \cup \\
 & t^e \cdot \lambda_{iR}
\end{tikzcd}$$

Now one has formula for $\dim X_w = \ell(w) \dots$

Hence $\mathcal{M}^{\mathrm{loc}}_{I}$ is too big, want to find closed subscheme which is flat $/ \mathcal{O}_E$. Let

K = Galois hull of $F/\mathbb{Q}_p$, let k' = res. field of $\mathcal{O}_K$.

Number the embeddings as $\varphi_1, \dots, \varphi_e$, hence get $r_1, \dots, r_e$.
$\Rightarrow K \supset E$

Splitting model over Spec $\mathcal{O}_K$: repr. functor on $(\mathrm{Sch}/\mathcal{O}_K)$: $S \mapsto$

$$\begin{array}{ccccccccc}
\Lambda_{t_0} \otimes \mathcal{O}_S & \longrightarrow & \Lambda_{t_1} \otimes \mathcal{O}_S & \longrightarrow & \cdots & \longrightarrow & \Lambda_{t_{m-1}} \otimes \mathcal{O}_S & \longrightarrow & \Lambda_{t_0} \otimes \mathcal{O}_S \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^e & \longrightarrow & \mathcal{F}_1^e & \longrightarrow & & \longrightarrow & \mathcal{F}_{m-1}^e & \longrightarrow & \mathcal{F}_0^e \\
\cup & & \cup & & & & \cup & & \\
\mathcal{F}_0^{e-1} & \longrightarrow & \mathcal{F}_1^{e-1} & \longrightarrow & & \longrightarrow & & \longrightarrow & \\
\cup & & \cup & & & & & & \\
\vdots & & \vdots & & & & & & \\
\cup & & & & & & & & \\
\mathcal{F}_0^1 & & & & & & & & \\
\cup & & & & & & & & \\
\mathcal{F}_0^0 = (0) & & & & & & & &
\end{array}$$

where $\mathcal{F}_i^j$ are $\mathcal{O}_F \otimes \mathcal{O}_S$-submodules s.t.

- $\mathcal{F}_i^j$ loc. direct summand as $\mathcal{O}_S$-module, of rank $\sum_{l=1}^{j} r_l$
- $(\pi \otimes 1 - 1 \otimes \varphi_j(\pi))(\mathcal{F}_i^j) \subset \mathcal{F}_i^{j-1}$, $\forall i, j$

This repres. by proj. scheme $\widetilde{\mathcal{M}}^{\mathrm{loc}}_{I}$ over Spec $\mathcal{O}_K$.

Proposition: The map $\{\mathcal{F}_i^j\}_{i \in I, j=1,\dots,e} \mapsto \{\mathcal{F}_i^e\}_{i \in I}$ induces a $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K$ - morphism

$$\widetilde{\mathcal{M}}^{\mathrm{loc}}_{I} \longrightarrow \mathcal{M}^{\mathrm{naive}}_{I} \otimes_{\mathcal{O}_E} \mathcal{O}_K$$

which is an isomorphism on generic fiber.

Definition : The local model $M^{loc}(G,\mu)_K$ is the image of the composed morphism (closed subscheme)

$$\tilde{M}^{loc}_I \longrightarrow M^{naive}_I \otimes_{O_E} O_K \longrightarrow M^{naive}_I .$$

Theorem: M^{loc} is flat over O_E, with reduced special fiber. All irreducible components of special fiber are normal and with rational singularities. Under $i : M^{loc} \otimes_{O_E} k \hookrightarrow \mathcal{F}$ have

$$M^{loc} \otimes_{O_E} k = \bigcup_{Adm(\mu)} X_w ,$$

where $\mu = \mu_1 + \dots + \mu_e$.

Ingredients of proof: • Have unramified local models for base field K instead of $\mathbb{Q}_p$, and cocharacter $\mu_1, \dots, \mu_e \rightsquigarrow$

$$M^1_I , \dots , M^e_I \quad \text{over } O_K : \text{ flat by Görtz.}$$

Then

$$\tilde{M}_I = M^1_I \tilde{\times} M^2_I \tilde{\times} \dots \tilde{\times} M^e_I , \quad \text{i.e. have}$$

diagram of phs under $\mathcal{G}^2 \times \dots \times \mathcal{G}^e$

$$\begin{tikzcd} & \tilde{M}^1_I \arrow[dl] \arrow[dr] & \\ \prod_{l=1}^{e} M^l_I & & \tilde{M}_I \end{tikzcd}$$

Hence $\tilde{M}_I$ is flat.

- Next, use Görtz's proof to show that $\overline{M}^{loc}$ is reduced, with all irreduc. comp. normal w. rational singul.
- Determine the extreme elements in $\overline{M}^{loc} = \bigcup X_w$:

math. Archive ← Haines / Ngo

Görtz, home page ←
- Finally, show that generic point of irreducible comp. can be lifted.

Finally, I want to show what the flatness conj. amounts to, in case $I = \{0\}$. Let

$$\widetilde{M}^{naive}(S) = \{ \mathcal{F} \in M^{naive}(S),\ \varphi : \mathcal{F} \xrightarrow{\sim} \mathcal{O}_S^r \}.$$

Get

$$\begin{tikzcd} \widetilde{M}^{naive} \arrow[d, "GL_r"'] \arrow[drr, "\varphi"] & & \\ M^{naive} & & \mathcal{N} \end{tikzcd}$$

Here

$$\mathcal{N}(R) = \{ A \in M_r(R);\ \det(T \cdot I - A) = \prod_{\varphi} (T - \varphi(\pi))^{r_\varphi},\ Q(A) = 0 \}.$$

Proposition : φ is smooth of relative dimension rd.

Note that the special fiber of $\mathcal{N}$ is

$$\overline{\mathcal{N}} = \{ A \in M_r ;\ char_A(T) = T^r,\ A^e = 0 \} \subset Nilp_r.$$

Conj. above is implied by the following conjecture on nilpotent matrices / arbitrary (alg. closed) field k.

<u>Conjecture</u>: $\bar{\mathcal{N}}$ is reduced.

true for char $k = 0$ (Weyman) and for $e = 2$ (Weyman)

and if $e \geq r$ (Kostant: the last eqn. is redundant)

The application is that when r_φ differ by at most 1, the special fiber is irreducible of same dim. as generic fiber. Hence if special fiber is reduced, then flatness.