

Lüning, July 00

Talk: On the structure of affine flag varieties for GL_n .

joint w. Pappas: ^{State the} establishing a link between affine Grassmann and nilp cone, with applications. In some sense converse to Lusztig.

k perfect field, Π indeterminate. Let $G = GL_n$. Affine Grassmannian

\mathcal{D}_G represents the following functor on k -algebras

$$R \mapsto \left\{ \begin{array}{l} \text{projective } R[\Pi]\text{-module } \mathcal{L} \text{ in } R[\Pi]^d \text{ s.t. } \exists n > 0 \\ \Pi^n \cdot R[\Pi]^d \subset \mathcal{L} \subset \Pi^n R[\Pi]^d \end{array} \right\}$$

\mathcal{L}_0

This is an ind-scheme, increasing union of closed subschemes as $n \rightarrow \infty$,
 via $\mathcal{L} \mapsto \bar{\mathcal{L}} \subset \Pi^n L_0/R / \Pi^n L_0/R: \Pi$ -stable, loc direct summand.

The loop group L^+G with $L^+G(R) = G(R[\Pi])$ acts transitively
 and \uparrow ind-group scheme

$$\mathcal{D}_G = L^+G / L^+G.$$

Here $\mathfrak{g} = L^+G$ integral loop group, $\mathfrak{g}(R) = G(R[\Pi])$. The
 orbits of \mathfrak{g} on \mathcal{D}_G are parametrized by dominant coweights of
 GL_n ,

$$\underline{s} = (s_1 \geq s_2 \geq \dots \geq s_n) \rightsquigarrow \mathcal{O}_{\underline{s}}$$

And $\dim \mathcal{O}_{\underline{s}} = \langle \underline{s}, 2\rho \rangle$ and $\mathcal{O}_{\underline{s}'} \subset \mathcal{O}_{\underline{s}}$ iff $\underline{s}' \leq \underline{s}$.

Now fix $e \geq 1$ and r with $0 \leq r \leq ed$. Consider the

following closed subscheme of \mathcal{G}_e , inlet r

$$X(r, e, d)(R) = \{ \mathcal{L} \in \mathcal{G}(R); \quad \pi^e L_0 \otimes R \subset \mathcal{L} \subset L_0 \otimes R; \}$$

$$\det \text{char } \pi | (\mathcal{L} / \pi^e L_0 \otimes R) \cong T^r \quad \}.$$

Then

$$X(r, e, d)_{\text{red}} = \overline{O}_{s_0} \quad \text{where } s_0 = (e, \dots, e, f, 0, \dots, 0)$$

$$r = ce + f, \quad 0 \leq f < e.$$

This can be identified with a closed subscheme of a Grassmannian

as follows: Let $(V_e, \pi_e) =$ standard VS of dim. e over k
 + regular nilpotent endo.

Let

$$(W, \pi) = (V_e, \pi_e)^d \quad \text{Then}$$

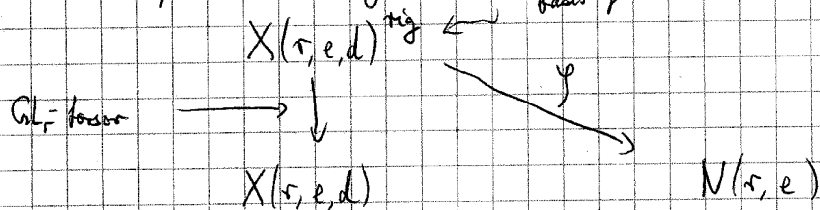
$$X(r, e, d)(R) = \{ \mathcal{F} \subset W \otimes_k R; \quad \mathcal{F} \text{ loc. on } \text{Spec } R \text{ direct sum,} \}$$

$$\left. \begin{array}{l} \mathcal{F} \text{ } \pi\text{-stable} \\ \det \text{char } \pi | \mathcal{F} \cong T^r \end{array} \right\}$$

closed subscheme of $\text{Grass}_r(W)$. Let

$$N(r, e) = \{ A \in M_r; \quad \text{char}_A \cong T^r, \quad A^e = 0 \}.$$

Now form diagram basis of \mathcal{F}



Note
 $N(r, e) = \overline{N}_{s_0}$

Diagram is equivariant w.r.t. $G \times GL_r$, where G acts through $GL_d(\mathbb{C}[T]/T^e)$ and 1st factor on the source, 2nd factor on target only.

Main Theorem: The morphism γ is smooth of relative dim. $r-d$.

About the proof: There exists a unique minimal stratum in $X(r, e, d)$, check by explicit matrix calculation smoothness along this stratum.

Remarks: (i) Let $e=d=r$. Then $N = Nilp_d$ and $\dim X = \dim N$. In this case have a natural section s to γ : Write $w \in W$ as

$$w = (v_0, v_1, \dots, v_{d-1}) \quad , \quad v_i \in V_d \quad , \quad s.t.$$

$$\pi w = (0, v_0, v_1, \dots, v_{d-2})$$

Now put $s(A) = F = \{(0, A^{d-2}v, \dots, Av, 0) : v \in V_d\}$

Then $\pi \circ s$ is an open immersion: $X(d, d, d)$ is a compactification of $Nilp_d$, due to Lusztig.

(ii) If replace W by $\bigoplus (V_{e_i}, \pi_{e_i})$, statement becomes wrong. Also, not a Thm. of reductive group theory, wrong for other groups.

Applic. to Sing.
of Schubert varieties

Corollary 1: Any (reduced) orbit closure \overline{O}_s is normal w. rational singularities. Its singularities are smoothly equivalent to singularities

occurring in nilpotent orbit closures for a general linear group

Proof: May assume that s effective, i.e. $s_i \geq 0$. Take $e \geq s_i$

and $r = \sum s_i \Rightarrow \overline{O}_s \subset X(r, e, d)$. Result follows from

corresp. result for nilpotent orbit closures (Mehrotra-van der Kallen).

Remark: Normality was studied by Matthei in context of Kac-Moody algebras. Recently Faltings proved normality for all gener. affine flag varieties, for any reductive group G/k , with G_{der} simply connected

\mathcal{D} not reduced. Want reduced subschemes

Conjecture: $N(\tau, e)$ is a reduced scheme // w/ orbit closure.

Note $N(\tau, e) = \overline{N_{\mathcal{D}_0}}$

Conject: True, if $\tau \leq e$.

Theorem (Weyman): The conjecture holds if $e=2$, and in general if $\text{char } k = 0$ //

Corollary 2: If conjecture holds, then $X(\tau, e, d)$ is reduced //

Corollary 3: For $n=1, 2, \dots$ let

$$X_n = \mathcal{L} \subset \mathcal{L} \subset \mathcal{D}; \quad \pi^n L_0 \subset \mathcal{L} \subset \pi^{-n} L_0, \quad \text{char } \frac{\pi L / \pi^n L_0}{\pi L / \pi^n L_0} \cong T^{nd}$$

Hence $X_1 \subset X_2 \subset \dots$ chain of closed subschemes of \mathcal{D} .

Corollary 3: If conjecture holds, then X_n is reduced and

$$\lim_{\rightarrow} X_n = L SL_d / L^+ SL_d //$$

Hence get explicit presentation of $L SL_d / L^+ SL_d$ as a limit of reduced schemes (it is known that this can be done, but not explicit).

Next want to relate 2 standard procedures of dealing with singularities of Schubert var. resp. w/ orbit closures.

Fix decomposition $\tau = (\tau_1, \dots, \tau_e)$ of τ with $0 \leq \tau_i \leq d$. μ_i

$$\mu_i = (1^{\tau_i}, 0^{d-\tau_i}) \quad \text{muscle coweight of } GL_d$$

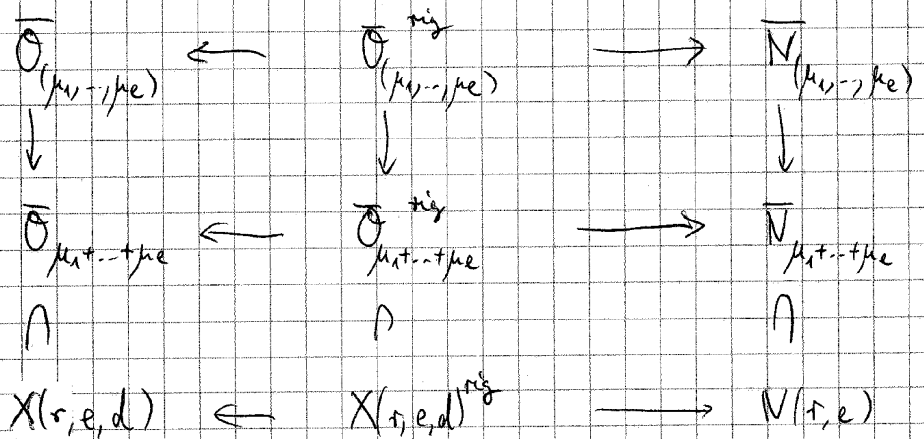
Then $\sum \mu_i$ has 2 interpretations:

- coweight of GL_d
- dual partition τ^\vee of τ

Correspondingly, get

$$\begin{cases} \overline{O}_{\mu_1 + \dots + \mu_e} \subset X(\tau, e, d) \\ \overline{N}_{\mu_1 + \dots + \mu_e} \subset N(\tau, e) \end{cases}$$

Corollary 4: We have a diagram with cartesian squares



Here $\overline{O}_{(\mu_1, \dots, \mu_e)} = \overline{O}_{\mu_1} \hat{\times} \dots \hat{\times} \overline{O}_{\mu_e}$ is the Demazure resolution (= convolution) and $\overline{N}_{(\mu_1, \dots, \mu_e)}$ the Springer resolution → Polo's talk

Other applications

Now let us pass to full flag variety (something similar can be done for partial flag varieties). It represents the functor

$$F_G: R \mapsto \{ (L_0, L_1, \dots, L_{d-1}) \in \mathcal{D}^d; L_0 \subset L_1 \subset \dots \subset L_{d-1} \subset L^d = \mathbb{A}^d \text{ s.t. } L_i/L_{i-1} \text{ loc free of rank } 1, \forall i \}$$

Then $F_G = LG/B$ B Borel subgroup

Thomson (Goetz): For any $v \in \tilde{W}$, the orbit closure \bar{O}_v is normal, with rational singularities.

We finally give an application to some closed subvarieties of F_G .

Fix r with $0 \leq r \leq d$. Consider standard chain in $k((\pi))^d$,

$$L_0 \subsetneq L_1 \subsetneq \dots \subsetneq L_{d-1} \subsetneq L_d = \pi L_0$$

Consider the closed subscheme $\mathcal{M}(r, d)$ of F_G consisting of (L_0, \dots, L_{d-1}) s.t.

$$\begin{array}{ccccccc} L_0 \otimes R & \rightarrow & L_1 \otimes R & \rightarrow & \dots & \rightarrow & L_{d-1} \otimes R \rightarrow L_d \otimes R \\ \cup & & \cup & & & & \cup \\ L_0 & \rightarrow & L_1 & \rightarrow & \dots & \rightarrow & L_{d-1} \rightarrow L_d \\ \cup & & \cup & & & & \cup \\ \pi L_0 \otimes R & \rightarrow & \pi L_1 \otimes R & \rightarrow & \dots & \rightarrow & \pi L_{d-1} \otimes R \rightarrow \pi L_d \otimes R \end{array}$$

s.t. $L_i / \pi L_i \otimes R$ is a π -module of rank r , $\forall i$

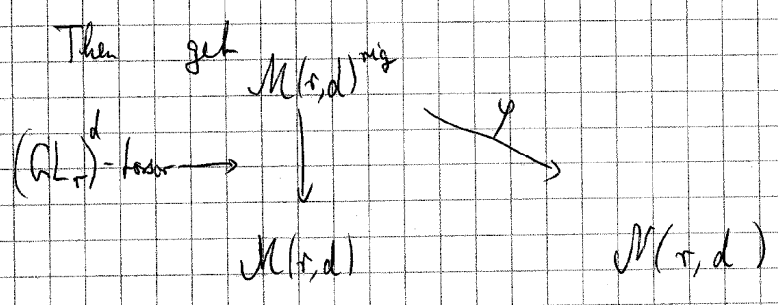
So, we are considering flags very close to standard flag.

Equivalently,

$$\left\{ \begin{array}{ccccccc} V_0 & \rightarrow & V_1 & \rightarrow & \dots & \rightarrow & V_{d-1} \rightarrow V_0 \\ \cup & & \cup & & & & \cup \\ F_0 & \rightarrow & F_1 & \rightarrow & \dots & \rightarrow & F_{d-1} \rightarrow F_0 \end{array} \right\}$$

all cyclic compos. are zero.

Let $\mathcal{M}(r, d) = \{ (A_0, \dots, A_{d-1}) \in M_r^d; A_0 A_1 \dots A_{d-1} = A_1 A_2 \dots A_{d-1} A_0 = \dots = 0 \}$



0 for d

kind of sing.
Easier than on \mathbb{P}^n ,
but less a resolution

Theorem (Faltings): The morphism γ is smooth of relative dim. $r+d$.

Theorem (Göttsche): $\mathcal{M}(r,d)$ is reduced, its irreducible components are normal with rational singularities.

Remarks: (i) Hesse $\mathcal{M} = \bigcup \mathcal{O}_w$ (Kottwitz-R)
 $w \in \mathbb{C}_{w_0}(\mu_r)$,
 some $w_0 \in S_d$

(ii) \mathcal{M} is Cohen-Macaulay for $d \leq 5$, and probably in general.

(iii) For $r \leq 3$, Faltings has a "modification" of $\mathcal{M}(r,d)$ with toroidal singularities.

(iv) Göttsche's problem ~~analogous~~ for a symplectic group. Uses normality of Schubert varieties (Faltings)

Recover classical results on variety of ^{circular} complexes.