Luminy, July 00

Talk: On the structure of affine flag varieties for GL_d.

joint w. Pappas: State thm establishing a link between affine Grassm. and nilp. cone, with applications. In some sense converse to Lusztig.

k perfect field, Π indeterminate. Let $G = GL_d$. Affine Grassmannian

$\mathcal{Q}_G$ represents the following functor on k-algebras

$R \mapsto \{$ projective $R[[\Pi]]$-module $\mathcal{L}$ in $R((\Pi))^d$ s.t. $\exists\, n > 0$

$$\Pi^n \cdot \underset{L_0}{\underset{\shortparallel}{R[[\Pi]]^d}} \subset \mathcal{L} \subset \Pi^{-n} R[[\Pi]]^d \quad \}$$

This is an ind-scheme, increasing union of closed subschemes as $n \to \infty$, via $\mathcal{L} \mapsto \bar{\mathcal{L}} \subset \Pi^{-n} L_0 \otimes R / \Pi^n L_0 \otimes R$: Π-stable, loc. direct summand.

The loop group LG with $LG(R) = G(R((\Pi)))$ acts trans. and

ind-group scheme

$$\mathcal{Q}_G = LG / L^+G.$$

Here $\mathcal{G} = L^+G$ integral loop group, $\mathcal{G}(R) = G(R[[\Pi]])$. The orbits of $\mathcal{G}$ on $\mathcal{Q}_G$ are parametrized by dominant coweights of GL_d,

$$\underline{s} = (s_1 \geq s_2 \geq \dots \geq s_d) \mapsto \mathcal{O}_{\underline{s}}.$$

And $\dim \mathcal{O}_{\underline{s}} = \langle \underline{s}, 2\rho \rangle$ and $\mathcal{O}_{\underline{s}'} \subset \bar{\mathcal{O}}_{\underline{s}}$ iff $\underline{s}' \leq \underline{s}$.

Now fix $e \geq 1$ and r with $0 \leq r \leq ed$. Consider the following closed subscheme of $\mathcal{D}_{\mathcal{L}}$,

$$X(r,e,d)(R) = \{ \mathcal{L} \in \mathcal{D}(R);\ \Pi^e L_0 \otimes R \overset{\text{index } r}{\subset} \mathcal{L} \subset L_0 \otimes R;$$

$$\det \mathrm{char}_{\Pi | (\mathcal{L}/\Pi^e L_0 \otimes R)} \equiv T^r \}.$$

Then

$$X(r,e,d)_{red} = \overline{O}_{s_0} \quad \text{where} \quad s_0 = (e, \dots, e, f, 0, \dots 0)$$

$$r = ce + f, \quad 0 \leq f < e.$$

<u>This can be identified with a closed subscheme of a Grassmannian</u>

<u>as follows:</u> Let (V_e, Π_e) = standard VS of dim. e over k + regular nilpotent endo.

Let

$$(W, \Pi) = (V_e, \Pi_e)^d. \quad \text{Then}$$

$$X(r,e,d)(R) = \{ \mathcal{F} \subset W \otimes_k R;\ \mathcal{F} \text{ loc. on Spec } R \text{ direct summ.},$$

$$\mathcal{F} \ \Pi\text{-stable}$$

$$\mathrm{char}_{\Pi | \mathcal{F}} \equiv T^r \}.$$

Closed subscheme of $\mathrm{Grass}_r(W)$. Let

$$N(r,e) = \{ A \in M_r;\ \mathrm{char}_A \equiv T^r,\ A^e = 0 \}.$$

Now form diagram

basis of $\mathcal{F}$

$$\begin{tikzcd} X(r,e,d)^{\mathrm{rig}} \arrow[d, "\mathrm{GL}_r\text{-torsor}"'] \arrow[dr, "\varphi"] & \\ X(r,e,d) & N(r,e) \end{tikzcd}$$

Note
$N(r,e) = \overline{N}_{s_0}$

Diagram is equivariant w.r.t. $\mathcal{G} \times GL_r$, where $\mathcal{G}$ acts through $GL_d(k[\Pi]/\Pi^e)$ and 1st factor on the source, 2nd factor on target only.

Main Theorem: The morphism φ is smooth of relative dim. rd.

About the proof: There exists a unique minimal stratum in $X(r,e,d)$, check by explicit matrix calculation smoothness along this stratum.

Remarks: (i) Let $e=d=r$. Then $N = \mathrm{Nilp}_d$ and $\dim X = \dim N$. In this case have a natural section s to φ: Write $w \in W$ as

$$w = (v_0, v_1, \dots, v_{d-1}) \quad , v_i \in V_d \quad , \text{s.t.}$$

$$\Pi w = (0, v_0, v_1, \dots, v_{d-2})$$

Now put $s(A) = \mathcal{F} = \{(0, A^{d-2}v, \dots, Av, v) ; v \in V_d\}$

Then $\pi \circ s$ is an open immersion: $X(d,d,d)$ is a compactific. of Nilp_d, due to Lusztig.

(ii) If replace W by $\bigoplus (V_{e_i}, \Pi_{e_i})$, statement becomes wrong. Also, not a Thm. of reductive group theory, wrong for other groups.

Applic. to Sing. of Schubert varieties

Corollary 1. Any (reduced) orbit closure $\overline{O}_{\underline{s}}$ is normal w. rational singularities. Its singularities are smoothly equivalent to singularities occurring in nilpotent orbit closures for a general linear group.

Proof: May assume that $\underline{s}$ effective, i.e. $s_d \geq 0$. Take $e \geq s_1$ and $r = \sum s_i \Rightarrow O_{\underline{s}} \subset X(r,e,d)$. Result follows from corresp. result for nilpotent orbit closures (Mehta - van der Kallen).

<u>Remark:</u> Normality was studied by Mathieu in context of Kac-Moody algebras. Recently Faltings proved normality for all gener. affine flag varieties, for any reductive group G/k, with G_{der} simply connected.

$\mathcal{D}$ not reduced. Want reduced subschemes

<u>Conjecture:</u> $N(r,e)$ is a reduced scheme.

Note $N(r,e) = \overline{N}_{\lambda_0} \leftarrow$ nilpotent orbit closure.

<u>Kostant:</u> True, if $r \leq e$.

<u>Theorem (Weyman):</u> The conjecture holds if $e = 2$, and in general if $\operatorname{char} k = 0$.

<u>Corollary 2:</u> If conjecture holds, then $X(r,e,d)$ is reduced.

Corollary 3: For $n = 1, 2, \ldots$ let

$$X_n = \{ \mathcal{L} \subset \mathcal{D} \; ; \; \Pi^n L_0 \subset \mathcal{L} \subset \Pi^{-n} L_0, \; \operatorname{char}_{\Pi \mid \mathcal{L}/\Pi^n L_0} \equiv T^{nd} \}.$$

Hence $X_1 \subset X_2 \subset \ldots$ chain of closed subschemes of $\mathcal{D}$.

<u>Corollary 3:</u> If conjecture holds, then X_n is reduced and

$$\varinjlim X_n = LSL_d / L^+SL_d .$$

Hence get explicit presentation of LSL_d / L^+SL_d as a limit of reduced schemes (it is known that this can be done, but not explicit).

Next want to relate 2 standard procedures of dealing with singularities of Schubert var. resp. nilp. orbit closures.

Fix decomposition $\underline{r} = (r_1, \ldots, r_e)$ of r with $0 \le r_i \le d$. $\rightsquigarrow$

$$\mu_i = (1^{r_i}, 0^{d-r_i}) \qquad \text{minuscule coweight of } GL_d.$$

<u>Then</u> $\Sigma_i \mu_i$ <u>has 2 interpretations</u>: coweight of GL_d / dual partition $\underline{r}^\vee$ of r

Correspondingly, get $\overline{O}_{\mu_1+\cdots+\mu_e} \subset X(r,e,d)$ / $\overline{N}_{\mu_1+\cdots+\mu_e} \subset N(r,e)$

<u>Corollary</u> 4: We have a diagram with cartesian squares

$$\begin{array}{ccccc}
\overline{O}_{(\mu_1,\ldots,\mu_e)} & \longleftarrow & \overline{O}^{\,\mathrm{rig}}_{(\mu_1,\ldots,\mu_e)} & \longrightarrow & \overline{N}_{(\mu_1,\ldots,\mu_e)} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{O}_{\mu_1+\cdots+\mu_e} & \longleftarrow & \overline{O}^{\,\mathrm{rig}}_{\mu_1+\cdots+\mu_e} & \longrightarrow & \overline{N}_{\mu_1+\cdots+\mu_e} \\
\cap & & \cap & & \cap \\
X(r,e,d) & \longleftarrow & X(r,e,d)^{\mathrm{rig}} & \longrightarrow & N(r,e)
\end{array}$$

Here $\overline{O}_{(\mu_1,\ldots,\mu_e)} = \overline{O}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \overline{O}_{\mu_e}$ is the Demazure resolution (= convolution) and $\overline{N}_{(\mu_1,\ldots,\mu_e)}$ the Springer resolution. Polo's talk

Other applications

Now let us pass to full flag variety (something similar can be done for partial flag variétés). It represents the functor

$$\mathcal{F}_G : R \longmapsto \{ (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{d-1}) \in \mathcal{D}^d ;\ \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{d-1} \subset \mathcal{L}_d = \pi^{-1}\mathcal{L}_0$$
$$\text{s.t. } \mathcal{L}_i/\mathcal{L}_{i-1} \text{ loc. free of rank } 1, \forall i \}.$$

Then $\mathcal{F}_G = LG/\mathcal{B}$ $\qquad \mathcal{B}$ Iwahori subgroup.

[illegible]

Theorem (Goertz): For any $w \in \widetilde{W}$, the orbit closure $\overline{O}_w$ is normal, with rational singularities.

We finally give an application to some closed subvarieties of $\mathcal{F}_G$.

Fix r with $0 \le r \le d$. Consider standard chain in $k((\Pi))^d$,

$$L_0 \subsetneq L_1 \subsetneq \dots \subsetneq L_{d-1} \subsetneq L_d = \Pi^{-1} L_0$$

Consider the ~~closed~~ subscheme $M(r,d)$ of $\mathcal{F}_G$ consisting of $(\mathcal{L}_0, \dots, \mathcal{L}_{d-1})$ s.t.

$$\begin{tikzcd}
L_0\otimes R \arrow[r] & L_1 \otimes R \arrow[r] & \cdots \arrow[r] & L_{d-1}\otimes R \arrow[r] & L_d \otimes R \\
\mathcal{L}_0 \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \mathcal{L}_1 \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \cdots \arrow[r] & \mathcal{L}_{d-1} \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \mathcal{L}_0 \arrow[u, phantom, sloped, "\subset"] \\
\Pi L_0 \otimes R \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \Pi L_1 \otimes R \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \cdots \arrow[r] & L_{d-1}\otimes R \arrow[r] \arrow[u, phantom, sloped, "\subset"] & L_d \otimes R \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

s.t. $\mathcal{L}_i / \Pi L_i \otimes R$ proj. R-module of rank r, $\forall i$

So, we are considering flags very close to standard flag.

Equivalently,

all cyclic compos. are zero.

$$\left\{\begin{tikzcd}
V_0 \arrow[r] & V_1 \arrow[r] & \cdots \arrow[r] & V_{d-1} \arrow[r] & V_0 \\
\mathcal{F}_0 \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \mathcal{F}_1 \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \cdots \arrow[r] & \mathcal{F}_{d-1} \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \mathcal{F}_0 \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}\right\}$$

Let $N(r,d) = \{ (A_0, \dots, A_{d-1}) \in M_r^d ;\ A_0 A_1 \cdots A_{d-1} = A_1 A_2 \cdots A_{d-1} A_0 = \cdots = 0 \}$

Then get

$$\begin{tikzcd}
 & M(r,d)^{rig} \arrow[d, "(GL_r)^d\text{-torsor}"'] \arrow[dr, "\varphi"] & \\
 & M(r,d) & N(r,d)
\end{tikzcd}$$

Kind of sing.

Easier than our thm, but less a reduction

__Theorem__ (Faltings): The morphism φ is smooth of relative dim. rd.

__Theorem__ (Görtz): $\mathcal{M}(r,d)$ is reduced, its irreducible components are normal with rational singularities.

__Remarks__: (i) Have $$\mathcal{M} = \bigcup_{\substack{w \le t_{w_0(\mu_r)},\\ \text{some } w_0 \in S_d}} \mathcal{O}_w \qquad \text{(Kottwitz - R).}$$

(ii) $\mathcal{M}$ is Cohen-Macaulay for $d \le 5$, and probably in general.

(iii) For $r \le 3$, Faltings has a „modification" of $\mathcal{M}(r,d)$ with toroidal singularities.

(iv) Görtz's proved analogue for symplectic group. Uses normality of Schubert varieties (Faltings).

Recover classical results on variety of circular complexes.