THE PRINCIPAL BLOCK

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Abstract. In this talk for winter 2022/2023 Arithmetische Geometrie Oberseminar in Bonn, we review the representation theory of the principal block of a reductive group over a nonarchimedean local field. This block is equivalent to the module category for the so-called Iwahori-Hecke algebra, whose algebraic structure we recall.

Notation

Let F be a nonarchimedean local field with ring of integers O and residue field $\mathbb{F}_q$. In these notes, the coefficient field for all representations and modules is $\mathbb{C}$. Starting from Section 2, G denotes a split reductive O-group, B a Borel subgroup, U the unipotent radical, and T a maximal split torus of B.

1. Basics of smooth representation theory

In this section only, let G be a locally profinite group. Let V be a representation of G. A vector $v \in V$ is smooth if it is fixed by a compact-open subgroup of G, and the representation V is smooth if each of its elements is smooth. In topological language, V is smooth if and only if the action map $G \times V \to V$ is continuous for the discrete topology on V. Let $\mathrm{Mod}(G)$ denote the category of smooth G-modules.

The usual operations of representation theory—contragredient, tensor product, inflation, restriction, induction, and so on—all carry over to the smooth category. We highlight two important differences.

First, the definitions of some of these operations, such as induction and contragredient, must be modified from their naive definitions by restricting to the smooth vectors. For example, the contragredient in the smooth category is defined to be

$$V^\vee := \mathrm{Hom}_G(V, \mathbb{C})^\infty,$$

where W^∞ denotes the smooth vectors in a representation W of G.

Second, in the smooth category there are two kinds of induction. Let H be a closed subgroup of G and $\rho\colon H \to \mathrm{GL}(W)$ a smooth representation.

- Smooth induction $\mathrm{Ind}_H^G(W)$, defined as the smooth functions $f\colon G \to W$ such that $f(hx) = \rho(h)f(x)$ for all $x \in G$ and $h \in H$. The group G acts in the usual way, by right translation. In other words, as an exemplar of the previous point,

$$\mathrm{Ind}_H^G(W) := \left(\mathrm{Ind}_{H_{\mathrm{disc}}}^{G_{\mathrm{disc}}}(W)\right)^\infty$$

  where the subscript "disc" denotes the discrete topology.
- Compact induction $\text{c-Ind}_H^G(W)$, the subspace of $\mathrm{Ind}_H^G(W)$ consisting of those functions f such that $\mathrm{supp}(f)$, which is a union of right H-cosets, is compact in $H\backslash G$.

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When G is discrete, the functors Ind_H^G and Res_H^G are adjoint to each other in both ways. But in the smooth settings, the two adjunctions separate: here the adjoint pairs are $(\mathrm{Res}_H^G, \mathrm{Ind}_H^G)$ and $(\text{c-}\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$.

When G is discrete, in particular, finite, representations of G are the same as $\mathbb{C}[G]$-modules. This equivalence has an analogue for locally profinite groups in which the group ring $\mathbb{C}[G]$ is shrunk to account for the topology on G.

Definition 1. The Hecke algebra $\mathcal{H}(G)$ of G is the algebra $C_c^\infty(G)$ of smooth, compactly-supported, complex-valued functions, and in which multiplication is defined by convolution:

$$(f_1 * f_2)(x) := \int_G f_1(y) f_2(y^{-1}x)\,\mathrm{d}y.$$

Remark 2. To avoid the choice of Haar measure in the definition of the Hecke algebra, one can instead define the Hecke algebra as the convolution algebra of smooth compactly-supported distributions. A choice of Haar measure identifies this version of the Hecke algebra with our version.

The Hecke algebra is noncommutative if G is, but even worse, it is unital only when G is discrete. Indeed, the unit should be the Dirac delta distribution, but this distribution is not a function unless G is discrete. There is an adequate replacement, however: a rich supply of idempotents. For us, the most important are the weighted indicator functions

$$e_K := \frac{\mathbb{1}_K}{\mathrm{vol}(K)}$$

of compact-open subgroups K, where $\mathbb{1}_X$ denotes the indicator function of X. We will use them to show that nondegenerate $\mathcal{H}(G)$-modules are the same as smooth G-modules.

Lemma 3. *Let V be a smooth representation of G. There is a unique action of $\mathcal{H}(G)$ on V such that*

$$\langle fv, v^\vee \rangle = \int_G f(x) \langle xv, v^\vee \rangle \,\mathrm{d}x$$

for all $f \in \mathcal{H}(G)$ and smooth linear functionals $v^\vee \colon V \to \mathbb{C}$.

Proof. The test function f is a finite sum of indicator functions of left cosets of compact-open subgroups of G. Refining the decomposition if necessary, we can assume that these sets are all cosets for the same subgroup K, which fixes V. Now define

$$\mathbb{1}_{gK} * v := \mathrm{vol}(K) gv. \tag{1}$$

We omit the necessary check that the resulting definition is independent of all choices. □

The module of Lemma 3 has a special property: it is nondegenerate in the sense that each of its elements is fixed by some idempotent. Using (1) again, we can recover a smooth representation of G from such an $\mathcal{H}(G)$-module.

Corollary 4. *There is a canonical equivalence of categories between smooth representations of G and nondegenerate $C_c^\infty(G)$-modules.*

We will henceforth pass freely between smooth representations of G and nondegenerate $\mathcal{H}(G)$-modules.

Let K be a compact open subgroup of G. We now restrict our attention to smooth representations V of G that are generated by their K-fixed vectors V^K. Let $\mathrm{Mod}_K(G)$

denote the category of such representations. The algebra that controls $\mathrm{Mod}_K(G)$ is the K-spherical Hecke algebra

$$\mathcal{H}(G,K) := e_K\mathcal{H}(G)e_K = C_c^\infty(K\backslash G/K).$$

Unlike the full Hecke algebra, the K-spherical Hecke algebra is unital: the unit is e_K.

However, the relationship between $\mathrm{Mod}_K(G)$ and $\mathrm{Mod}\big(\mathcal{H}(G,K)\big)$ is not so straightforward as one might first think. In one direction, given a $\mathcal{H}(G,K)$-module E, one forms the smooth G-representation

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} E =: M_K(E).$$

In the other direction, given a smooth representation V of G, the fixed-points space

$$V^K = e_K V =: m_K(V).$$

is a $\mathcal{H}(G,K)$-module. All in all, we have an adjoint pair (M_K, m_K) of functors

$$\mathrm{Mod}\big(\mathcal{H}(G,K)\big) \underset{m_K}{\overset{M_K}{\rightleftarrows}} \mathrm{Mod}_K(G).$$

Furthermore, the fixed-points functor m_K is exact because

$$V = e_K V \oplus (1-e_K)V, \tag{2}$$

and this functor is a left quasi-inverse to M_K:

$$\mathrm{id} \xrightarrow{\sim} m_K \circ M_K, \qquad \text{that is,} \qquad E \xrightarrow{\sim} \big(\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} E\big)^K.$$

It follows formally that M_K is fully faithful [Sta22, Tag 0FWV].

So m_K and M_K are very close to being quasi-inverses. It can even be shown, using Zorn's Lemma, that m_K and M_K induce a bijection on irreducible objects. Nevertheless, these functors are not equivalences in general.

Example 5. Let G be the F-points of a reductive F-group and let B be a Borel subgroup. Consider the short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow C^\infty(B\backslash G) \longrightarrow V \longrightarrow 0$$

of representations of G, in which the first map is inclusion of constant functions. Let K be a maximal compact subgroup of G satisfying the Iwasawa decomposition $G = BK$. Then clearly $C^\infty(B\backslash G)^K = \mathbb{C}$, so that $V^K = 0$ by exactness of K-fixed points. Now consider the dual sequence

$$0 \longrightarrow V^\vee \longrightarrow C^\infty(B\backslash G)^\vee \longrightarrow \mathbb{C} \longrightarrow 0.$$

Let W be the G-subspace of $C^\infty(B\backslash G)^\vee$ generated by the K-fixed vectors. Since K-fixed points commutes with the contragredient, $(V^\vee)^K = 0$ and $W \neq 0$. The morphism

$$W \to \mathbb{C} \tag{3}$$

in $\mathrm{Mod}_K(G)$ becomes an isomorphism after applying m_e, so if m_e were an equivalence then (3) would have to be an isomorphism, meaning that $W \simeq \mathbb{C}$. On the other hand, using Frobenius reciprocity and the fact that

$$C^\infty(B\backslash G)^\vee \simeq \mathrm{Ind}_B^G(\delta_B^{-1}),$$

where $\delta_B\colon B \to \mathbb{C}$ is the modulus character, it can be shown that $\mathbb{C}$ does not embed into $C^\infty(B\backslash G)^\vee$, a contradiction.

This example simplifies quite a bit when $G = \mathrm{GL}_2(F)$, in which case V is an irreducible representation called the Steinberg representation which we discuss in Section 3.

Proposition 6. *The following are equivalent.*

1. *The functors M_K and m_K are quasi-inverses, so that* $\mathrm{Mod}_K(G) \simeq \mathrm{Mod}\big(\mathcal{H}(G,K)\big)$.
2. $\mathrm{Mod}_K(G)$ *is closed in* $\mathrm{Mod}(G)$ *under subobjects.*
3. $\mathrm{Mod}_K(G)$ *is a Serre subcategory of* $\mathrm{Mod}(G)$.

Proof. Given (1), let $V \in \mathrm{Mod}_K(G)$ and let W be a G-stable subspace of V. Replacing W by $\mathcal{H}(G)e_K W$, we may assume that $W^K = 0$. Now the projection map $V \to V/W$ becomes an isomorphism after applying m_K, and since m_K is an equivalence by assumption, $W = 0$ after all.

Conversely, given (2), showing (1) amounts to showing that the co-unit

$$(M_K \circ m_K)(V) \to V,$$

which is surjective because $V \in \mathrm{Mod}_K(G)$, is also injective. Indeed, its kernel has trivial K-fixed points, so by (2), the kernel is trivial.

The equivalence of (2) and (3) is an easy exercise. □

Remark 7. In our application to the principal block, an even stronger conclusion holds: the functor M_K realizes $\mathrm{Mod}_K(G)$ as a direct summand of $\mathrm{Mod}(G)$. It seems that this stronger conclusion does not hold in the generality of Proposition 6, and it would be interesting to construct a counterexample.

2. The principal block

We now return to the setting of reductive groups over a local field, specializing the discussion of the previous section to a particularly important compact-open subgroup of $G(F)$.[1]

Definition 8. The standard Iwahori subgroup I of $G(F)$ is the pre-image of $B(\mathbb{F}_q)$ in $G(O)$:

$$\begin{array}{ccc} I & \longrightarrow & G(O) \\ \downarrow & \lrcorner & \downarrow \\ B(\mathbb{F}_q) & \longrightarrow & G(F) \end{array}$$

An Iwahori subgroup of $G(F)$ is any $G(F)$-conjugate of the standard Iwahori.

Remark 9. Our choice of standard Iwahori depends on the choice of Borel subgroup B and integral model G of G_F. But for our purposes it is enough to work with a single Iwahori, and we will suppress the adjective "standard".

Example 10. One Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ is the set of matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ of the form

$$\begin{bmatrix} a & b \\ pc & d \end{bmatrix}, \qquad a, b, c, d \in \mathbb{Z}_p.$$

More generally, one Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ is the set of matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$ all of whose entries below the diagonal are divisible by p.

[1] In the rest of the article, we abuse notation by writing $\mathrm{Mod}(G)$ for $\mathrm{Mod}(G(F))$ and $\mathcal{H}(G)$ for $\mathcal{H}(G(F))$.

The goal of this section is to prove that the Iwahori subgroup satisfies the equivalent properties of Proposition 6. To make the proof more elementary, we will actually prove something a bit weaker by restricting our attention to admissible representations. Although the stronger statement holds for all smooth representations, I'm not sure how to prove it without using the theory of the Bernstein decomposition, which felt to me like too large of a black box. We will use a superscript "adm" to denote the admissible representations of G or finite-dimensional $\mathcal{H}(G, I)$-modules.

The key to the proof is the following lemma.

Lemma 11. *Let $V \in \mathrm{Mod}_I^{\mathrm{adm}}(G)$. Suppose that*

$$\text{every nonzero smooth subrepresentation of } V \text{ has an } I\text{-fixed vector.} \tag{$*$}$$

Then any exact sequence of smooth admissible representations

$$1 \longrightarrow V \longrightarrow V \longrightarrow V'' \longrightarrow 1$$

with $V''^I = 0$ splits.

Before discussing the proof of Lemma 11, we explain the main consequence of this lemma. In the proof, we construct a cousin of the functor M_I which is related to M_I in roughly the same way as the functor c-Ind is relatd to Ind. We use duality and admissibility to pass between the functors, both of which have their own advantages.

Corollary 12. *The functor M_I realizes $\mathrm{Mod}_I^{\mathrm{adm}}(G)$ as a direct summand of $\mathrm{Mod}^{\mathrm{adm}}(G)$.*

We call $\mathrm{Mod}_I(G)$ the principal block of $\mathrm{Mod}(G)$.

Proof that Lemma 11 implies Corollary 12. There are several steps.

First, we claim that if $V \in \mathrm{Mod}_I^{\mathrm{adm}}(G)$ satisfies ($*$) then each of its subrepresentations W lies in $\mathrm{Mod}_I(G)$. This follows by applying the conclusion of the lemma to the exact sequence obtained from inclusion into W of the subrepresentation generated by W^I.

Second, given $E \in \mathrm{Mod}\big(\mathcal{H}(G, I)\big)$, consider the space

$$M_I'(E) := \{\text{smooth } \phi\colon G(F)/I \to E : \phi * f = f \cdot \phi\},$$

where $f \cdot \phi$ denotes the action of f on E. Then $M_I'(E)$ is a representation of G by left translation. We claim that $M_I'(E)$ satisfies ($*$). Indeed, any subrepresentation V contains a function ϕ such that $\phi(1) \neq 0$, and then a direct computation shows that $(e_I * \phi)(1) \neq 0$, so that $0 \neq e_I * \phi \in V^I$.

Third, since

$$M_I(E) := \mathcal{H} \otimes_{\mathcal{H}(G,I)} E = \mathcal{H} e_I \otimes_{\mathcal{H}(G,I)} E = C_c(G(F)/I) \otimes_{\mathcal{H}(G,I)} E,$$

there is a natural map

$$M_I(E) \to M_I'(E), \tag{4}$$

which becomes the identity on E after taking I-fixed points. A priori (4) is neither injective nor surjective, we claim that it is an isomorphism when $\dim E < \infty$. Lemma 11 shows that (4) is surjective, in other words, that $M_I'(E)$ is generated by its I-fixed vectors. For injectivity, use the fact that $M_I'(E^\vee)$ is isomorphic to $M_I(E)^\vee$.

Fourth, combining the previous steps, we see that when $\dim E < \infty$, every subrepresentation of $M_I(E)$ lies in $\mathrm{Mod}_I(G)$.

Finally, let $V \in \mathrm{Mod}_I^{\mathrm{adm}}(G)$ be arbitrary. The action map $M_I(V^I) \to V$ is surjective by assumption. Any subrepresentation of V is a quotient of a subrepresentation of $M_I(E)$, and $\mathrm{Mod}_I(G)$ is closed in $\mathrm{Mod}(G)$ under quotients. So we are done. □

The proof of Lemma 11 is of a similar flavor to the proof given above. In the interest of brevity, we will not repeat it. Instead, we merely isolate one notable input to the proof.

As a preliminary step, let's recall two important operations in the smooth representation theory of reductive groups: parabolic induction and the Jacquet functor.

First, given a character χ of $T(F)$, the parabolic induction of χ to G is the induced representation $\mathrm{Ind}_B^G \chi$, where we inflate χ to $B(F)$ via the surjection $B \to T$. Second, the Jacquet module of a representation V of G is the space V_U of coinvariants of the unipotent radical U. By Frobenius reciprocity, $((-)_U, \mathrm{Ind}_B^G(-))$ is an adjoint pair.

Lemma 13. *The canonical map $V^I \to (V_U)^{T(O)}$ is an isomorphism.*

This map is induced by the projection $V \to V_U$.

Proof sketch. Both parts of the proof are classical results of harmonic analysis. In the interest of brevity, we sketch the proof of surjectivity. A proof of injectivity may be found in [Blo97].

Let $\bar{U}$ be the opposite parabolic of U with respect to T, let $U_0 = I \cap U(F) = U(O)$, and let $\bar{U}_0 = I \cap \bar{U}(F)$, the pre-image of 1 in $\bar{U}(O)$ under the reduction map $U(O) \to U(\mathbb{F}_q)$. The key input to surjectivity is that I decomposes as a set-theoretic product

$$I \simeq \bar{U}_0 \times T(O) \times U_0.$$

More precisely, the multiplication map from the righthand side to the lefthand side is a homeomorphism. Such a decomposition holds for many other compact-open subgroups, and is called a Iwahori factorization in that generality.

to do!!!! □

Remark 14. We know from Example 5 that some part of the proof of Corollary 12 must break down if I is replaced with a maximal special compact subgroup K. Every step of our proof of Corollary 12 (modulo Lemma 11) would work just as well for K in place of I. But Lemma 13 is one place where the proof would fail for K, since K does not admit an Iwahori factorization.

Corollary 15. *A smooth irreducible representation V of G lies in the principal block if and only if it is a subquotient of $\mathrm{Ind}_B^G(\chi)$ for some unramified character χ of $T(F)$.*

Proof. By Frobenius reciprocity,

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\chi)) \simeq \mathrm{Hom}_T(V_U, \chi) \simeq \mathrm{Hom}_T\big((V_U)_{T(O)}, \chi\big).$$

It follows from (2) that the canonical map $V^I \to V_I$ is an isomorphism, so that $(V_U)_{T(O)} \simeq V^I$ by Lemma 13. Hence

$$\mathrm{Hom}_T(V_U, \chi) \simeq \mathrm{Hom}_T\big(V^I, \chi\big).$$

Since V is irreducible, it lies in the principal block if and only if $V^I \neq 0$, and the left hom set above has dimension 0 or 1. If it has dimension 1 for some χ, meaning that V embeds in $\mathrm{Ind}_B^G(\chi)$, then $V^I \neq 0$. Conversely, if $V^I \neq 0$ then there is some χ such that the right hom set above is nonzero, so that V embeds in $\mathrm{Ind}_B^G(\chi)$. □

3. Example: GL_2

In this section we list, without proof, the irreducible representations in the principal block of GL_2. Here the torus T is the diagonal matrices, the Borel B is the upper-triangular matrices, and the modulus character δ_B is

$$\left[\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right] \mapsto |a/b|.$$

We start with a preliminary observation. In general, the group of characters χ of $F^\times$ acts on representations of $\mathrm{GL}_2(F)$ by determinantal twisting:

$$\pi \mapsto (\chi \circ \det) \otimes \pi := \chi\pi$$

This action restricts to an action on the principal block of the unramified characters of $F^\times$. So the representations in our list can be freely twisted by unramified characters.

The construction of the list proceeds, via Corollary 15, by analyzing parabolic inductions of unramified characters of the maximal torus of diagonal matrices. Most of the time the parabolic inductions are irreducible, but on occasion they decompose into two pieces.

A more specific statement is supplied by the following proposition. Let $w = \left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]$ be the Weyl element. Given a character χ of $T(F)$ with

$$\chi \colon \left[\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right] \mapsto \chi_1(a)\chi_2(b),$$

the Weyl conjugate ${}^w\chi$ is the character

$${}^w\chi \colon \left[\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right] \mapsto \chi_2(a)\chi_1(b).$$

The Steinberg representation of $\mathrm{GL}_2(F)$ is the quotient

$$\mathrm{St} := C^\infty(B(F)\backslash G(F))/\mathbb{C},$$

where $\mathbb{C}$ is realized as constant functions. This representation is discrete series, but not supercuspidal.

Proposition 16.

(1) Two parabolic inductions $i_B^G\chi$ and $i_B^G\sigma$ share an irreducible subquotient if and only if $\chi = \sigma$ or $\chi = {}^w\sigma$, in which case they have the same composition factors.[2]

(2) $i_B^G\chi$ is reducible if and only if $\chi = \delta_B^{\pm 1/2}(\phi \circ \det)$ for some character ϕ of $F^\times$, in which case it has length two and its composition factors are ϕ and $\phi \cdot \mathrm{St}$.

In fact, we can do a bit better and draw a picture of the situation. The unramified characters of $T(F)$ are of the form

$$\chi_{s,t} \colon \left[\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right] \mapsto s^{\mathrm{ord(a)}} \cdot t^{\mathrm{ord(b)}}$$

where $s, t \in \mathbb{C}^\times$. The Weyl element acts on $\chi_{s,t}$ by swapping s and t. The determinantal characters lie on the diagonal $s = t$, the modulus character is $\delta_B = \chi_{q^{1/2},q^{-1/2}}$, and its unramified twists form the locus $st^{-1} = q^{\pm 1}$. Once we identify χ with its parameter, part (1) of Proposition 16 shows that the irreducible representations of the principal block are fibered over the variety $(\mathbb{C}^\times)^2/S_2$, depicted in Figure 1. Over most of this variety, the gray locus in the picture, the fibers are singletons. But over the red lines $s = q^{\pm 1}t$ the fibers are two elements, twists of $\{\mathrm{id}, \mathrm{St}\}$.

[2]Here we abuse notation by writing i_B^G for $i_{B(F)}^{G(F)}$.

The set $\mathrm{Irr}\big(\mathrm{Mod}_K(G)\big)$ of irreducible representations in the principal block admits a topology called the Jacobson topology. The resulting space is non-Hausdorff: its universal Hausdorff quotient is $(\mathbb{C}^\times)^2/S_2$, and the locus where the space fails to be Hausdorff is precisely the locus where the fibration fails to be one-to-one.

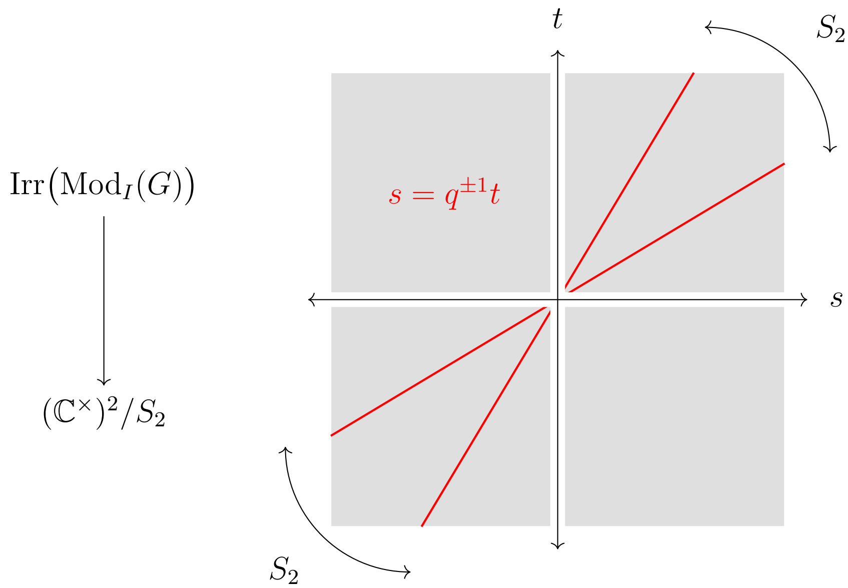

FIGURE 1. The space of principal supercuspidal supports for $\mathrm{GL}_2(F)$

4. Iwahori-Hecke algebras

Our goal in this section is to understand the algebraic structure of the ring that controls the principal block, namely $\mathcal{H}(G, I)$.

The first observation is that this ring has as a basis the set of indicator functions $\mathbb{1}_{IgI}$ of double cosets of I in G. These double cosets are conveniently parameterized, and we can describe (generators for) the relations between them.

The parameterization of double cosets is called the (affine) Bruhat decomposition. In this decomposition, a certain group called the affine Weyl group plays a starring role.

Definition 17. The affine Weyl group W_a of G is the quotient

$$W_a := N_G(T)(F)/T(O),$$

where $N_G(T)$ is the normalizer of T in G.

The Bruhat decomposition asserts that the double cosets of I in G are parameterized by W_a. In more detail, given $w \in W_a$ and a lift $\dot{w}$ of w to $N_G(T)(F)$, the resulting double coset $I\dot{w}I$ is independent of the choice of $\dot{w}$, and we write $IwI := I\dot{w}I$.

Theorem 18 (Bruhat decomposition)**.** *The assignment $w \mapsto IwI$ defines a bijection*

$$W_a \simeq I\backslash G/I.$$

We may therefore index a basis of $\mathcal{H}(G, I)$ by W_a, via the indicator functions $\mathbb{1}_{IwI}$. Our next step is to describe some relations between these basis elements. For this, we must first recall more information about the structure of affine Weyl groups.

The coarse structure is simple: the affine Weyl group is a split extension of the ordinary Weyl group by a lattice.

Proposition 19. *$W_a \simeq X_*(T) \rtimes W$.*

Proof sketch. The affine Weyl group fits into the short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T(F)/T(O) & \longrightarrow & W_a & \longrightarrow & W & \longrightarrow & 1. \\ & & \simeq \big\downarrow \text{val} & & & & & & \\ & & X_*(T) & & & & & & \end{array}$$

In fact, this short exact sequence splits. The splitting follows from the theory of the Tits lift [Tit66]: in brief, the failure of the projection $N_G(T)(F) \to W$ to split is measured by a two-cocycle of W valued in the order-two elements of $T(F)$. Since these all lie in $T(O)$, this projection splits after we mod out by $T(O)$. □

Remark 20. The isomorphism of Proposition 19 is not canonical: in the language of Bruhat-Tits theory, it depends on a choice of special point in the apartment of T.

An important feature of ordinary Weyl groups is that they are Coxeter groups. This feature is almost shared by affine Weyl groups, but there is a small complication that arises when G is not simply-connected.

Our choice of Borel subgroup B produces a set S of simple reflections in W making the pair (W, S) a Coxeter system. Let I be the set of irreducible components of the root system of G. For $i \in I$, let $\tilde{\alpha}_i$ be the longest root of the ith component, let

$$\tilde{s}_i = (-\widetilde{\alpha}_i^\vee, s_{\widetilde{\alpha}_i})$$

in the semidirect product description of W_a, and let $S_a := S \cup \{\tilde{s}_i \mid i \in I\}$.

Fact 21. *If G is simply connected then the pair (W_a, S_a) is a Coxeter system with Coxeter diagram the extended Coxeter diagram of G.*

When G is not simply-connected, we gather information about W_a by comparing it to the affine Weyl group W_a^{sc} of the simply-connected cover G^{sc} of the derived subgroup of G. Recall that when G is simply connected, $X_*(T)$ is the coroot lattice $\mathbb{Z}R^\vee$. Let

$$\pi_1(G) := X_*(T)/\mathbb{Z}R^\vee,$$

the algebraic fundamental group of G.

Proposition 22. *$W_a \simeq W_a^{\text{sc}} \rtimes \pi_1(G)$.*

Proof. W_a^{sc} is a normal subgroup of W_a since the Weyl group normalizes the root lattice. □

Example 23. When $G = \mathrm{GL}_2$,

- $W_a \simeq \mathbb{Z}^2 \rtimes S_2$, where S_2 acts on $\mathbb{Z}^2$ by permutation, and
- the Coxeter generators of W_a are $s = (0, \sigma)$ and $t = (e_2 - e_1, \sigma)$.

Although W_a is not quite a Coxeter group, we can nonetheless equip it with a length function by stipulating that the non-Coxeter elements have length zero.

Definition 24. The length function $\ell\colon W_a \to \mathbb{N}$ is defined by

$$\ell(w, \sigma) := \ell^{\text{sc}}(w)$$

where $w \in W_a^{\text{sc}}$ and $\sigma \in \pi_1(G)$, we use the identification of Proposition 22, and $\ell^{\text{sc}}\colon W_a^{\text{sc}} \to \mathbb{N}$ is the length function of the Coxeter system (W_a^{sc}, S_a).

We can now state the fundamental relations between the basis elements $\mathbb{1}_{IwI}$ of $\mathcal{H}(G, I)$.

Definition 25. The universal Hecke algebra $\mathcal{H}$ for G is the $\mathbb{C}[z, z^{-1}]$-algebra generated by the symbols $\{e_w : w \in W_a\}$ satisfying the relations

$$e_s e_w = \begin{cases} e_{sw} & \text{if } \ell(sw) > \ell(w) \\ z e_{sw} + (z-1) e_w & \text{if } \ell(sw) < \ell(w). \end{cases} \tag{5}$$

Let $\mathcal{H}_q$ be the algebra obtained from $\mathcal{H}$ by specializing z to q.

Proposition 26. *Normalize the Haar measure on G so that* $\mathrm{vol}(I) = 1$*. Then the assignment $e_w \mapsto \mathbb{1}_{IwI}$ extends to an algebra isomorphism*

$$\mathcal{H}_q \simeq \mathcal{H}(G, I).$$

Proof. Let $\phi_w := \mathbb{1}_{IwI}$ for brevity. The exchange condition for Coxeter groups [Bou68, IV§1.5] implies that if $\ell(sw) < \ell(w)$ then there is some reduced decomposition for w starting with s. From this, it suffices by induction on the length of w to show that

(1) $\phi_s * \phi_s = (q-1)\phi_s + \phi_1$.
(2) if $\ell(sw) > \ell(w)$ then $\phi_s * \phi_w = \phi_{sw}$

The first of these is an easy calculation. As for the second, it is easy to see that $\phi_s * \phi_w = c\phi_{sw}$ for some integer $c \geq 1$. To show that $c = 1$, use that integration over G is a homomorphism $\varepsilon\colon \mathcal{H} \to \mathbb{C}$. Letting

$$q_w := \mathrm{vol}(IwI) = [I : wIw^{-1}],$$

we see that $c = q_s q_w / q_{sw}$.[3] But one can show by a calculation with cosets that $q_{sw} \leq q_s q_w$. Hence $c = 1$. □

Earlier we saw two semidirect product decompositions of the affine Weyl group, Propositions 19 and 22. Each decomposition gives rise to a related decomposition of the Iwahori-Hecke algebra: roughly speaking, we replace abelian groups by their group rings, Weyl groups by their Hecke algebras, and semidirect product by twisted tensor product.

In what follows, the notation $A \mathbin{\widetilde{\otimes}} B$ denotes a twisted tensor product of $\mathbb{C}$-algebras in which A and B embed as subalgebras but the relations between A and B must be further specified.

From Proposition 26, we immediately deduce the first decomposition.

Proposition 27. *Let $\mathcal{H}^{\text{sc}}$ denote the universal Hecke algebra of G^{sc}. Then there is an algebra isomorphism*

$$\mathcal{H} \simeq \mathcal{H}^{\text{sc}} \mathbin{\widetilde{\otimes}} \mathbb{C}[\pi_1(G)]$$

where multiplication in the algebra on the righthand side is defined by

$$e_w e_\sigma = e_{\sigma(w)} e_w \qquad (w \in W_a^{\text{sc}}, \sigma \in \pi_1(G)).$$

[3] In fact, $q_w = q^{\ell(w)}$. This reflects the nature of the cells IwI as the $\mathbb{F}_q$-points of an affine variety.

The second decomposition, arising from the description $W_a \simeq X_*(T) \rtimes W$, will permit us to compute the center of $\mathcal{H}$. To start with, we explain how $\mathbb{C}[X_*(T)]$ is a subalgebra of $\mathcal{H}$.

Direct computation shows that $\ell(\lambda) = 2\langle\lambda, \rho\rangle$ for $\lambda \in X_*(T)$ dominant. Now given an arbitrary $\lambda \in X_*(T)$, choose a decomposition $\lambda = \mu - \nu$ with μ and ν dominant. The length formula above implies that the element

$$x^\lambda := q^{-\langle\lambda,\rho\rangle} e_\mu e_\nu^{-1}$$

is independent of the choices of μ and ν. The same formula also shows that

$$x^\lambda x^\mu = x^{\lambda+\mu}.$$

Hence the assignment $\lambda \mapsto x^\lambda$ defines an injective ring map $\mathbb{C}[X_*(T)] \to \mathcal{H}$.

Fact 28. *Let $\mathcal{H}_{\mathrm{fin}}$ denote the Hecke algebra of the finite Weyl group W.*[4] *Then*

$$\mathcal{H} \simeq \mathbb{C}[X_*(T)] \mathbin{\widetilde{\otimes}} \mathcal{H}_{\mathrm{fin}}$$

where the algebra on the righthand side admits $\mathbb{C}[X_*(T)]$ and $\mathcal{H}_{\mathrm{fin}}$ as subalgebras and satisfies the additional relations

$$x^\lambda e_s - e_s x^{s(\lambda)} = (q-1)\frac{x^\lambda - x^{s(\lambda)}}{1 - x^{-\alpha_s^\vee}} \qquad (s \in S, \lambda \in X_*(T))$$

with $\alpha_s^\vee$ is the simple coroot corresponding to s.

From Fact 28 one can deduce the structure of the center of the Hecke algebra.

Corollary 29. $Z(\mathcal{H}) \simeq \mathbb{C}[z, z^{-1}]\big[X_*(T)\big]^W$.

We can now conclude that the principal block is a true block in the sense that it cannot be further subdivided.

Corollary 30. *The principal block is indecomposable.*

Proof. The center of the principal block is the center of $\mathcal{H}(G, K)$, which is an indecomposable ring. But if the principal block decomposed as a direct sum of two subcategories then its center would decompose into the centers of the factors. ☐

Guide to the literature

In this section we point the curious reader to places in the literature where the ideas of this article are discussed more fully.

Basic facts about smooth representation theory are reviewed in many places, including Cartier's Corvallis article [Car79], Bernstein's lecture notes [BR92], and Ngô Bảo Châu's lecture notes [NB16]. Renard's beautiful book [Ren10] provides exhaustive details. Our discussion of the relationship between $\mathrm{Mod}_K(G)$ and $\mathrm{Mod}\big(\mathcal{H}(G, K)\big)$ draws on the influential article of Bushnell and Kutzko on type theory [BK98], of which the theory of the principal block is a special case.

The Iwahori subgroup is named after Nagayoshi Iwahori, who studied it in an important article with Matsumoto [IM65]. Its basic theory has been explained in many places, among them Tits's Corvallis article [Tit79] and the forthcoming book of Kaletha and Prasad. The observation that the category of smooth representations generated by their Iwahori-fixed

[4] In other words, $\mathcal{H}_{\mathrm{fin}}$ is the subalgebra of $\mathcal{H}$ generated by $\{e_s : s \in S\}$.

vectors constitutes a block in the full category of smooth representations is originally due to Borel [Bor76], and our proof sketch draws on Section 4 of his article.

The standard source for the representation theory of GL_2 is the book of Bushnell and Henniart [BH06], which carefully describes in a famous exercise of Bourbaki [Bou68, Ex. 24 to IV§2].

The decomposition of Fact 28 is due to Lusztig, but I read about it in a nice article of Heckman and Opdam on Hecke algebras [HO97].

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