1 Monadic theories

Let $\mathscr{L}$ a first-order language and X an $\mathscr{L}$-structure. Its monadic second-order theory can be defined by the following equivalent (at least for decidability and definability purposes) descriptions:

- Take the first-order theory of X and add an additional sort for its power set 2^X, together with a relation symbol $\in$ on $X \times 2^X$, to be interpreted in the obvious way.
- Assume without loss of generality $\mathscr{L}$ relational (replace functions by their graphs). It is the first-order theory of the Boolean algebra 2^X, adding for each relation symbol $R(X_1, \ldots, X_n)$ of $\mathscr{L}$ an relation symbol interpreted by "all X_i are singletons $\{x_i\}$ and $R(x_1, \ldots, x_n)$ holds".

In fact, we will be mostly interested in the case $\mathscr{L} = \{<\}$ and often replace 2^X by a sub-Boolean algebra $\mathcal{B}$ (or even a subposet) in the above. For getting used to monadic theories, we prove some easy results, which motivate our main result. In the following, all structures are regarded as monadic second-order structures, and definability is without parameters, unless mentioned otherwise.

Contrast the increasing expressive power of $(\mathbb{N}, x \mapsto x+1), (\mathbb{N}, <), (\mathbb{N}, +), (\mathbb{N}, +, \cdot)$ in the first-order case with the following results:

Proposition 1. *$(\mathbb{N}, <)$ and $(\mathbb{N}, x \mapsto x+1)$ are interdefinable.*

Proof. Defining $\bullet + 1$ from $<$ works as in the first-order case. For the converse note that $x \leq y$ if and only if there is a set $A \subseteq \mathbb{N}$ with $x, y \in A$, such that $a \in A \setminus \{y\}$ implies $a+1 \in A$, but $y+1 \notin A$, and $a \in A \setminus \{x\}$ implies $a \neq 0$ and $a-1 \in A$, but $x = 0$ or $x-1 \notin A$. □

We will see decidability of this theory later.

Lemma 2. *The first-order theory of $(\mathbb{N}, +, |)$ defines multiplication.*

Proof. It suffices to define squaring as $2xy = (x+y)^2 - x^2 - y^2$. But $x^2 = \operatorname{lcm}(x, x+1) - x$ as $x, x+1$ are always coprime. □

Lemma 3. *In a monoid $(X, \cdot)$, the map sending $x \in X$ to the submonoid $\langle x \rangle$ generated by x is definable.*

Proof. Being a submonoid is definable, and $\langle x \rangle$ is the smallest submonoid containing x. □

Proposition 4. *$(\mathbb{N}, +)$ defines multiplication.*

Proof. By lemma 2 it suffices to define divisibility. For this we may use lemma 3, as $x \mid y$ if and only if y lies in the additive submonoid generated by x. □

Furthermore note that $(\mathbb{N}, +, \cdot)$ defines a pairing function (that is, an injection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$), whence its monadic second-order theory defines the full second-order theory.

Proposition 5. *$(\mathbb{R}, +)$ interprets $(\mathbb{N}, +)$.*

Proof. Given a parameter $x \neq 0$, use lemma 3 to define $x\mathbb{N}$. Now any sentence $\varphi(\mathbb{N})$ can be reduced to $\forall x \neq 0 \varphi(x\mathbb{N})$ (or $\exists x \neq 0 \varphi(x\mathbb{N})$). □

To summarize: Over the naturals, successor or equivalently order are tractable, but addition is not. Over the reals, additions is still intractable. How about order?

In general, it turns out to be undecidable. One can interpret natural numbers (more precisely, elements of a countable model of Robinson arithmetic) by certain Cantor sets in $\mathbb{R}$, coding their behavior into a single parameter, which is defined by transfinite recursion ([GS82]; under the continuum hypothesis, this was already proven in [She75]). This suggests that positive results may be obtained by restricting the class $\mathcal{B}$ of sets.

It is known (and proven by automata-theoretic methods [Rab69]) that restricting $\mathcal{B}$ to F_σ-sets, that is, countable unions of closed sets, yields a decidable theory. Our main results are:

Theorem 6. *The monadic second-order theory of* $(\mathbb{R}, <)$ *with quantification restricted to Borel sets is decidable.*

Theorem 7. *Under ZF+Determinacy+Dependent Choice, the monadic second-order theory of* $(\mathbb{R}, <)$ *is decidable.*

As a final bit of context, let us relate order and topology.

Proposition 8. *On* $\mathbb{R}$*, the monadic-second order theory of order is biinterpretable with the theory of the Boolean algebra* $2^{\mathbb{R}}$ *with a predicate for open sets, or equivalently closed sets or the interior or closure operator.*

Proof. Interpreting the different topological notions with each other is trivial, as is defining topology by order. But topology also allows recovering order (up to reversing the order, or more precisely the relation $x < z < y \vee y < z < x$), as a subset is an interval if and only if it is connected in the topological sense. $\square$

To contrast our results above, the theory of topology of $\mathbb{R}^n$ for $n \geq 2$ is undecidable as it interprets the naturals numbers [Grz51], and the proof goes through even if we restrict to small classes of sets (like Boolean combinations of cubes). The main idea is that we may interpret natural numbers n by finite sets of n points and express for example equality by connecting the points pairwise with paths.

2 Descriptive Set Theory

In the following, many statements are given without (complete) proof, but all proofs can be found in [Kec12].

A topological space is Polish if it is separable and completely metrizable. Equivalently, it is homeomorphic to a closed subspace of $\mathbb{R}^{\mathbb{N}}$. Examples of Polish spaces include $\mathbb{R}^n$, the Cantor set $2^{\mathbb{N}}$, the Baire space $\mathbb{N}^{\mathbb{N}} \cong \mathbb{R} \setminus \mathbb{Q}$, any separable Banach space, any countable ordinal with the order topology, or the space of compact subsets of $\mathbb{R}^n$ with the Vietoris topology, induced by the Hausdorff metric.

Proposition 9. *A subset of a Polish space is Polish with the subspace topology if and only if it is* G_δ*, that is, a countable intersection of open sets.*

Proof. We will only use sufficiency, and for this it suffices to show that countable intersections of Polish subspaces are Polish, which is formal, and that open subsets are Polish. For this note that closed subspaces are Polish with the same metric, and that for $U \subseteq X$ open there is a closed embedding $U \to X \times \mathbb{R}$ given by $x \mapsto \left(x, \frac{1}{d(x, \complement U)}\right)$. $\square$

Recall that a σ-algebra on a set X is a sub-Boolean algebra of 2^X that is stable under countable unions, and the σ-algebra generated by $\mathcal{A} \subseteq 2^X$ is the smallest σ-algebra containing $\mathcal{A}$. The Borel σ-algebra of a Polish space is the σ-algebra generated by the open subsets.

Recall that a point x in a topological space is isolated if $\{x\}$ is open, and a subset is perfect if it is nonempty, closed, and has no isolated points (in the subspace topology). In particular, the image of any embedding of $2^{\mathbb{N}}$ into a Hausdorff space is perfect.

Proposition 10 (Perfect set property)**.** *Let A be an uncountable Borel subset of a Polish space. Then there is a topological embedding* $2^{\mathbb{N}} \to A$.

Proof. One can show that the ambient space X has a finer topology such that A is clopen by Borel induction on A, whence we may assume $A = X$. Removing the union of all countable open sets, we may assume that X has no isolated points (this is the only place where uncountability is used).

Then one constructs by recursion a Cantor scheme, that is, an infinite binary tree of nonempty open subsets such that any two siblings are disjoint, and along each branch the sets are decreasing and the diameter in some complete metric tends to zero. Finally, map each branch (that is, element of the Cantor set) to the unique element of the intersection of all sets along the branch. $\square$

Proposition 11. *Any totally disconnected compact metric space without isolated points is homeomorphic to* $2^{\mathbb{N}}$*.*

Proof. Under Stone duality, they correspond to countable atomless Boolean algebras. And the theory of atomless Boolean algebras is $\aleph_0$-categorical (in fact, it is the Fraïssé limit of the finite Boolean algebras). □

Corollary 12. *Any totally disconnected compact metric space is the union of a Cantor set and a countable set.*

Recall that a subset of a topological space is nowhere dense if its closure has empty interior. It is easy to show that the nowhere dense sets form an ideal of sets, that is, are stable under finite unions and taking subsets.

A subset of a topological space is meager if it is a countable union of nowhere dense sets. It is comeager if its complement is meager.

Theorem 13 (Baire)**.** *Meager sets in complete metric spaces have empty interior.*

In $\mathbb{R}$, being meager and being a Lebesgue null set are different notions of being small, and none implies the other.

A subset of a Polish space has the Baire property if it is the symmetric difference of an open set and a meager set.

Proposition 14. *Borel sets have the Baire property.*

Proof. Since open sets have the Baire property, it suffices to show that the sets having the Baire property form a σ-algebra. □

The following property is why we prefer meager sets (resp. sets with the Baire property) over Lebesgue null sets (resp. Lebesgue measurable sets):

Proposition 15. *Let A have the Baire property. Then either A is meager, or there is a nonempty open U such that $A \cap U$ is comeager in U.*

Proof. Write $A = U \triangle M$ with U open, M meager. If $U = \emptyset$, A is meager, and else the second property holds for U. □

3 Miscellaneous

Theorem 16 (Ramsey)**.** *Let X infinite, C finite, $n \in \mathbb{N}$ and f be a map from the n-element subsets of X to C. Then X has an infinite subset Y that is homegeneous, that is, there is $c \in C$ such that for all distinct $y_1, \ldots, y_n \in Y$ we have $f(\{y_1, \ldots, y_n\}) = c$.*

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