Derived Equivalences and Tilting Complexes

Tashi Walde

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Betreuer: Prof. Dr. Catharina Stroppel

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Zusammenfassung

Ist eine abelsche Kategorie $\mathcal{A}$ gegeben, so ist oft die Untersuchung ihrer derivierten Kategorie $D^*(\mathcal{A})$ von Interesse. Wir konzentrieren uns auf den Fall, im dem $\mathcal{A}$ die Kategorie von Moduln über einem beliebigen (nicht notwendigerweise kommutativen) Ring ist. Man kann sich dann fragen, für welche zwei Ringe Λ und Γ die entsprechenden derivierten Kategorien $D^*(\Lambda)$ und $D^*(\Gamma)$ als triangulierte Kategorien äquivalent sind; man nennt solche Ringe **deriviert äquivalent**.

In dieser Arbeit soll Rickards Charakterisierung [Ric89] der derivierten Äquivalenz von Ringen vorgestellt und im Detail ausgearbeitet werden. Zentral für diese Charakterisierung ist die Definition von sogenannten **Tilting-Komplexen** über Λ; es handelt sich dabei um bestimmte Objekte in der derivierten Kategorie $D^*(\Lambda)$, die diese triangulierte Kategorie in einem gewissen Sinne „erzeugen“. Wir zeigen, dass zwei Ringe Λ und Γ genau dann deriviert äquivalent sind, wenn Γ als opponierter Endomorphismenring eines Tilting-Komplexes über Λ auftritt. Als Folgerung aus Rickards Satz beweisen wir anschließend, dass die Zentren deriviert äquivalenter Ringe isomorph sind.

Abschließend soll der Fall betrachtet werden, in dem ein Tilting-Komplex über Λ mit opponiertem Endomorphismenring Γ komponentenweise die Struktur eines Λ-Γ-Bimoduls trägt. Diese Betrachtung soll durch ein übersichtliches Beispiel motiviert werden und liefert dann in diesem Fall eine konkrete Beschreibung der Äquivalenz $D^*(\Gamma) \xrightarrow{\simeq} D^*(\Lambda)$ als Tensorprodukt mit dem Tilting-Komplex.

Dieser Arbeit liegt ein Appendix bei, in dem die zur Verständnis der Arbeit notwendigen Definitionen und Aussagen zu Kettenkomplexen, triangulierten Kategorien und derivierten Kategorien kurz zusammengefasst werden.

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1. Introduction

When studying rings and their representations it is natural to ask when two (different) rings have an equivalent module category. This question finds a satisfying answer in the following

Theorem (Morita)**.** *[DI71, Proposition 3.3] Two rings Λ and Γ are* ***Morita equivalent****, i.e. the module categories $\Lambda-\mathrm{Mod}$ and $\Gamma-\mathrm{Mod}$ are equivalent, if and only if Γ is isomorphic to the opposite endomorphism ring $\mathrm{End}_{\Lambda}^{\mathrm{opp}}(M)$ of some progenerator M of $\Lambda-\mathrm{Mod}$.* □

We call a Λ-module M a **progenerator** of the category $\Lambda-\mathrm{Mod}$ (which consists of all Λ-modules, not necessarily finitely generated) if M is projective, finitely generated and there is a Λ-epimorphism $M^{(I)} \twoheadrightarrow \Lambda$ for some (not necessarily finite) index set I.

Morita equivalence is a very strong notion and sometimes a weaker equivalence relation is more interesting. One can pass from the module category to its derived category (with the canonical triangulated structure) and replace Morita equivalence by so-called **derived equivalence**.
The derived category of an abelian category still carries a lot of information about the original category and, moreover, provides the natural framework for derived functors and homological algebra. Furthermore, many important invariants of a ring are invariant under derived equivalences; examples for this are the center (or more generally the Hochschild-cohomology), the Grothendieck group and the finiteness of global dimension.
It is therefore not surprising that mathematicians started investigating the conditions under which two rings can be derived equivalent. In their efforts, they searched for tools to systematically construct triangulated equivalences of derived categories.

An important milestone that should be mentioned here is the following

Theorem (Happel)**.** *[Hap88, §III, Theorem 2.10] Fix a field k and let Λ be a finite dimensional k-algebra of finite global dimension. Let M be a* ***tilting module**** *over Λ, i.e. a finitely generated Λ-module satisfying the following two conditions:*

TM1 *M has no higher self-extensions, i.e. $\mathrm{Ext}^i_{\Lambda}(M, M) = 0$ for $i > 0$ and*

TM2 *there is an exact sequence of the form*

$$0 \to \Lambda \to M_0 \to M_1 \to \cdots \to M_r \to 0,$$

where M_j is a summand of a sum of copies of M for all $j \in \{0, \dots, r\}$.

Then the algebras Λ and $\Gamma := \operatorname{End}_{\Lambda}^{\mathrm{opp}}(M)$ are bounded derived equivalent with respect to finitely generated modules, i.e. there is an equivalence of triangulated categories $D^b(\Lambda-\mathrm{mod}) \simeq D^b(\Gamma-\mathrm{mod})$. □

Happel's theorem is quite limited in scope and even when talking about specific kind of algebras it does not provide a complete characterization of the equivalence relation 'derived equivalence'.
There is a generalization and sharpening of Happel's result due to Rickard, which is the main object of this thesis. The idea is to keep the common underlying structure of the previous two theorems unchanged (we still compare Λ with the endomorphism ring of a suitable chosen object "over Λ") but to replace the notion of tilting *modules* by that of tilting *complexes*. A **tilting complex** is a bounded complex of finitely generated projective modules (hence an object in the derived category) which satisfies some conditions that generalize TM1 and TM2 above.

Once we have the correct definition of a tilting complex (which is introduced in Section 3), we will obtain the following

Theorem (Rickard)**.** *[Ric89, Theorem 6.4] Two rings Λ and Γ are derived equivalent if and only if Γ is isomorphic to the opposite endomorphism ring $\operatorname{End}_{D^b(\Lambda)}^{\mathrm{opp}}(T)$ of some tilting complex T over Λ.*

In the exposition of (the proof of) this theorem (Sections 2-5) we closely follow Rickard's original work as presented in the book by König and Zimmermann [KZ98, Chapter 3]. The proof that derived equivalent rings have isomorphic centers (Section 6) is taken from the same source [KZ98, Proposition 6.3.2].

There is a special situation when the equivalence $D^-(\Gamma) \xrightarrow{\simeq} D^-(\Lambda)$, which is associated by Rickard's theorem to a tilting complex T, can be described explicitly as a tensor product with T. In Section 7, first a concrete example and then a more general theorem describing this phenomenon will be presented.

*The notion of a 'tilting module' goes back to the theory quasi-hereditary algebras. Those algebras are finite dimensonal of finite global dimension and come along with two types of distinguished modules, called standard and costandard objects, respectively. In that setting, a tilting module is an object which has a filtration both by standard and by costandard objects, i.e. "it can be tilted and looks more or less the same again". In particular those tilting modules have no higher self-extensions, since one can show that there are no nontrivial higher extensions of a standard object by a costandard object. If one takes the direct sum of enough such tilting modules one can force condition TM2 to hold as well [KK99, Corollary 2.6]; in this way one produces tilting modules as in Happel's theorem.

The reader is expected to be familiar with basic notions concerning chain complexes, triangulated categories and derived categories. Appendix B provides a short introduction to all the concepts that are required in order to understand this thesis. At one point in the proof of Rickard's theorem (Section 5.2) we will have to deal with inverse limits in the homotopy category of chain complexes; this turns out to be more delicate than one might expect. We will have to make use of the so-called Mittag-Leffler condition for inverse systems; the relevant definitions and results can be found in Appendix C.
The terminology and notation we use is mostly standard. In any case, an overview over the recurring notation of this thesis is given in Appendix A.

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2. Bounded and unbounded derived categories

So far we have been somewhat imprecise with what we mean by *the* derived category. There are different derived categories which play a role for different applications; most importantly, a distinction can be made between bounded and unbounded derived categories. We always restrict at least to the right-bounded derived category $D^-(?)$ (which is equivalent to the homotopy category $K^-(?-\mathrm{Proj})$ of right-bounded chain complexes of projective modules [Wei94, Theorem 10.4.8]); we shall see now that for our purposes it does not matter whether one restricts further to the bounded derived category $D^b(?)$ or not. This investigation, also due to Rickard, produces an interesting characterization of the various subcategories of $D^-(?)$.

2.1. Intrinsic characterization of derived subcategories

Fix an unitary, associative (but not necessarily commutative) ring Λ. Let

$$D^b(\Lambda) \supset K^b(\Lambda-\mathrm{Proj}) \supset K^b(\Lambda-\mathrm{proj})$$

be the full subcategories of the derived category $K^-(\Lambda-\mathrm{Proj}) \simeq D^-(\Lambda)$, consisting of complexes of projective modules that are homology-bounded, resp. bounded, resp. bounded with finitely generated components.

These subcategories can be characterized using only the triangulated structure.

Theorem 2.1.1. *(i) An object X of $K^-(\Lambda-\mathrm{Proj})$ lies in $D^b(\Lambda)$ if and only if for all other objects Y of $K^-(\Lambda-\mathrm{Proj})$ we have:*

$$\forall n \ll 0\colon\ \mathrm{Hom}_{K^-}(Y, X[n]) = 0.$$

(ii) An object X of $D^b(\Lambda)$ is isomorphic to an object of $K^b(\Lambda-\mathrm{Proj})$ if and only if for all other objects Y of $D^b(\Lambda)$ we have:

$$\forall n \ll 0\colon\ \mathrm{Hom}_{K^-}(X[n], Y) = 0.$$

(iii) An object X in $K^b(\Lambda-\mathrm{Proj})$ is isomorphic to an object in $K^b(\Lambda-\mathrm{proj})$ if and only if the functor

$$\mathrm{Hom}_{K^b}(X, -) : K^b(\Lambda-\mathrm{Proj}) \longrightarrow \mathbf{AbGrp}$$

commutes with arbitrary direct sums.

Before we prove Theorem 2.1.1, we observe that it directly implies the following

Corollary 2.1.2. *Every derived equivalence of two rings which is defined on one of the levels* $D^-(?)$, $D^b(?)$, $K^b(?\text{−Proj})$ *or* $K^b(?\text{−proj})$ *restricts to a derived equivalence of all smaller levels.* □

2.2. Proof of the characterizaton

The remainder of Section 2 will be concerned with proving Theorem 2.1.1.

(i) Let objects X in $D^b(\Lambda)$ and Y in $K^-(\Lambda\text{−Proj})$ be given. We need to show $\mathrm{Hom}_{K^-}(Y, X[n]) = 0$ for all sufficiently small n. Without loss of generality, we may assume that Y is only nonzero in negative degrees. Since X has bounded homology, for all n sufficiently small the complex $X' := X[n]$ has nontrivial homology only in positive degrees. Now let X'' be the complex

$$\cdots \to X'_{-2} \to X'_{-1} \to \mathrm{Im}\left(X'_{-1} \to X'_0\right) \to 0 \to \cdots .$$

This complex has no homology and is therefore isomorphic to the zero complex in $D^-(\Lambda) \simeq K^-(\Lambda\text{−Proj})$. However, since $Y_i = 0$ for $i \geq 0$, every morphism $Y \to X'$ factors as $Y \to X'' \hookrightarrow X'$; hence every such morphism is zero.

Conversely, take some complex X with unbounded homology. We can then take infinitely many indices $i_0 > i_1 > i_2 > \cdots$ and elements $x_{i_j} \in \mathrm{Ker}\left(X_{i_j} \to X_{i_j+1}\right)$ that do not vanish in homology. This means, that for all $j \in \mathbb{N}$ the chain map $\lceil\Lambda\rfloor \to X[i_j]$ given in degree zero by $1 \mapsto x_{i_j}$ is nonzero in homology; hence $\mathrm{Hom}_{D^-}(\lceil\Lambda\rfloor, X[i_j]) \neq 0$. Finally, we choose $Y := \lceil\Lambda\rfloor$ and are done.

(ii) It is clear that if X is bounded we can shift it to the right, away from every right-bounded complex Y, so that their supports do not overlap anymore. This gives in particular that $\mathrm{Hom}_{D^b}(X[n], Y) = 0$ for all sufficiently small n.

To prove the converse, assume that X is a right-bounded complex of projective modules which has bounded homology but is not isomorphic to a bounded complex of projective modules.

First of all, we want to prove that there are infinitely many indices $n < 0$, such that $\mathrm{Im}(d_n : X_n \to X_{n+1})$ is not projective.
Assume on the contrary that there is an index N, such that for all $n \leq N$ the image of d_n is projective. Since X has bounded homology, we may also assume that there is no homology $H_n X$ for $n \leq N$. Hence, the short exact sequence

$$0 \to \mathrm{Ker}(d_n) \to X_n \xrightarrow{d_n} \mathrm{Im}(d_n) \to 0 \tag{2.1}$$

splits, since the rightmost term is projective. This gives a short exact sequence of complexes

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
\cdots & \longrightarrow & K_{N-3}\oplus K_{N-2} & \longrightarrow & K_{N-2}\oplus K_{N-1} & \longrightarrow & K_{N-1} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{d_{N-4}} & X_{N-3} & \xrightarrow{d_{N-3}} & X_{N-2} & \xrightarrow{d_{N-2}} & X_{N-1} & \xrightarrow{d_{N-1}} & X_N & \xrightarrow{d_N} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow d_{N-1} & & \| & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & K_N & \longrightarrow & X_N & \xrightarrow{d_N} & \cdots \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}
$$

where $K_n \coloneqq \mathrm{Ker}\,(d_n) = \mathrm{Im}\,(d_{n-1})$; the isomorphism $\mathrm{Ker}\,(d_n)\oplus\mathrm{Im}\,(d_n) \xrightarrow{\cong} X_n$ comes from the split short exact sequence (2.1). The upper-row complex is contractible (see Example B.1.7 in the Appendix); therefore the bottom row, which is a bounded complex of projective modules, is isomorphic to X in the homotopy category. This contradicts our assumption on X, so we see that there are indeed infinitely many indices $n < 0$ with $\mathrm{Im}(d_n)$ not projective.

Now we construct a complex Y in $K^-(\Lambda\text{−Proj})$ with bounded homology, such that $\mathrm{Hom}_{K^-}(X, Y[-n]) \neq 0$ for infinitely many $n < 0$. This will of course complete the proof.

So let $n < 0$ be such that $H_n X = 0$ and $\mathrm{Ker}\,(d_n) = \mathrm{Im}\,(d_{n-1})$ is not projective (by the previous result, there are infinitely many such n's). Let Y^n be the complex

$$\cdots \to X_{n-2} \to X_{n-1} \to X_n \to 0 \to \cdots ,$$

which arises by truncating X on the right, and consider the canonical map $f^n\colon X \to Y^n$.

If there was a homotopy $h\colon f^n \simeq 0$, then in particular we would have two maps $h_{n+1}\colon X_{n+1} \to X_n$ and $h_n\colon X_n \to X_{n-1}$ such that

$$\mathbb{1}_{X_n} = f^n_n = h_{n+1}d_n + d_{n-1}h_n.$$

Restricted to the submodule $\mathrm{Ker}\,(d_n) \subseteq X_n$ this equality would simply become $\mathbb{1}_{\mathrm{Ker}(d_n)} = d_{n-1}h_n$. This would mean precisely that h_n is a splitting of the surjection $X_{n-1} \overset{d_{n-1}}{\twoheadrightarrow} \mathrm{Im}\,(d_{n-1}) = \mathrm{Ker}\,(d_n)$. However, the non-projective $\mathrm{Ker}\,(d_n)$ would then be a summand of the projective X_{n-1}, which is a contradiction.

Therefore f^n is a nonzero element of $\mathrm{Hom}_{K^-}(X, Y^n)$.

To finish the construction, we can set $Y := \bigoplus_{n<0} Y^n[n]$, which is a complex of projective modules bounded to the right by the index 0 and inherits its bounded homology from X. We see that there are arbitrarily small n such that $\mathrm{Hom}_{K^-}(X, Y[-n])$ doesn't vanish, since this Hom-group contains $\mathrm{Hom}_{K^-}(X, Y^n)$ as a summand, which for the infinitely many $n < 0$ as above contains the nonzero element f^n.

(iii) Let X be a bounded complex of finitely generated Λ-modules. We need to show that for any collection $\{Z^i\}_{i\in I}$ of objects in $K^b(\Lambda\text{−Proj})$ the canonical injection

$$\begin{aligned} \bigoplus_{i\in I} \mathrm{Hom}\left(X, Z^i\right) &\hookrightarrow \mathrm{Hom}\left(X, \bigoplus_{i\in I} Z^i\right) \\ \left(\phi^i\right)_{i\in I} &\mapsto \left(t \mapsto \left(\phi^i(t)\right)_{i\in I}\right) \end{aligned}$$

is an isomorphism.

Take an element $\phi\colon t \mapsto (\phi^i(t))_{i\in I}$ of the right hand side. It has a preimage $(\phi^i)_{i\in I} \in \bigoplus_i \mathrm{Hom}(X, Z_i)$ if and only if almost all components ϕ^i are zero.

Since all components X_j of X are finitely generated, we can find finite sets $G_j \subset X_j$, such that ϕ^i is zero if and only if $\phi^i_j\big|_{G_j} = 0$ for all j where $X_j \neq 0$ (finitely many, since X is bounded). For each such j and each such $t_j \in G_j$ there are only finitely many $i \in I$ with $\phi^i_j(t_j) \neq 0$, since $\left(\phi^i_j(t_j)\right)_{i\in I}$ is an element in $\bigoplus_i Z^i_j$. So in total there are only finitely many indices $i \in I$ where $\phi^i \neq 0$.

For the converse, let X be a complex and assume that $\mathrm{Hom}_{K^b}(X, -)$ commutes with direct sums. Without loss of generality we can assume that X starts at 0, i.e. it is of the form

$$\cdots \to 0 \to X_0 \xrightarrow{d_0} X_1 \to \cdots \to X_n \to 0 \to \cdots$$

We will prove by induction on n that X is isomorphic in $K^b(\Lambda\text{−Proj})$ to a complex $\overline{X}$, which not only has finitely generated components but is also bounded in the same range as X, i.e. $\overline{X}_j$ is nonzero only for $0 \leq j \leq n$.

First of all we reduce to the case where X_0 is finitely generated.

By adding a trivial summand $X'_0 \xrightarrow{\mathbb{1}} X'_0$ to X, we may assume that X_0 is free, i.e. we can identify it with a module of the form $\Lambda^{(I)}$ for some index set I.

By the assumption on X we have an isomorphism

$$\mathrm{Hom}_{K^b}\left(X, \lceil \Lambda^{(I)} \rfloor\right) \cong \mathrm{Hom}_{K^b}\left(X, \lceil \Lambda \rfloor\right)^{(I)},$$

which means that the canonical map $X \to \lceil X_0 \rfloor \cong \lceil \Lambda^{(I)} \rfloor$ is homotopic to a map β which hits only finitely many components. Thus we can decompose X_0 as $\Lambda^{(J)} \oplus \Lambda^{(I \setminus J)}$ where J is finite and $\mathrm{Im}(\beta) \subseteq \Lambda^{(J)}$. Moreover the homotopy between $X \to \lceil X_0 \rfloor$ and β translates to a map $s\colon X_1 \to X_0$ with $\mathbb{1}_{X_0} - \beta = sd_0$. Postcomposing with the projection $p\colon X_0 \to \Lambda^{(I \setminus J)}$ and using $p\beta = 0$, we obtain $p = psd_0$. Therefore we have the following short exact sequence of complexes, which splits since the bottom nontrivial complex consists of free modules:

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & & & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\Lambda^{(J)} & \longrightarrow & \mathrm{Ker}(ps) & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_n \\
\downarrow & & \downarrow & & \| & & & & \| \\
X_0 & \xrightarrow{d_0} & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_n \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle ps} & & \downarrow & & & & \downarrow \\
\Lambda^{(I\setminus J)} & = & \Lambda^{(I\setminus J)} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & \\
0 & & 0 & & & & & &
\end{array}$$

Since $\Lambda^{(I \setminus J)} \xrightarrow{\mathbb{1}} \Lambda^{(I \setminus J)}$ is trivial in the homotopy category, we can replace X with the complex in the upper row, which still has projective components (as $\mathrm{Ker}(ps)$ is a direct summand of X_1) but now has finitely generated 0-th component. Clearly the range of nonzero entries has not widened during this process.

Now that we can assume X_0 to be finitely generated, let Y denote the truncated complex $X_1 \to \cdots \to X_n$. We have then a triangle $Y \to X \to \lceil X_0 \rfloor \rightsquigarrow$, so for any collection $(Z^i)_{i \in I}$ of objects in $K^b(\Lambda\text{–Proj})$ we get the following commutative diagram (of abelian groups) with exact columns:

$$\begin{array}{ccc}
\bigoplus_i \mathrm{Hom}(X, Z^i[1]) & \xrightarrow{\cong} & \mathrm{Hom}(X, \bigoplus_i Z^i[1]) \\
\downarrow & & \downarrow \\
\bigoplus_i \mathrm{Hom}(\lceil X_0 \rfloor, Z^i[1]) & \xrightarrow{\cong} & \mathrm{Hom}(\lceil X_0 \rfloor, \bigoplus_i Z^i[1]) \\
\downarrow & & \downarrow \\
\bigoplus_i \mathrm{Hom}(Y, Z^i) & \longrightarrow & \mathrm{Hom}(Y, \bigoplus_i Z^i) \\
\downarrow & & \downarrow \\
\bigoplus_i \mathrm{Hom}(X, Z^i) & \xrightarrow{\cong} & \mathrm{Hom}(X, \bigoplus_i Z^i) \\
\downarrow & & \downarrow \\
\bigoplus_i \mathrm{Hom}(\lceil X_0 \rfloor, Z^i) & \xrightarrow{\cong} & \mathrm{Hom}(\lceil X_0 \rfloor, \bigoplus_i Z^i)
\end{array}$$

Since $\lceil X_0 \rfloor$ is a complex of finitely generated projective modules, the functor

$\mathrm{Hom}\,(\lceil X_0 \rfloor, -)$ commutes with arbitrary direct sums by the 'only if' part of (iii), which we already proved. The functor $\mathrm{Hom}\,(X, -)$ also commutes with direct sums by assumption; hence the four outer vertical arrows in the diagram are isomorphisms, forcing $\bigoplus_i \mathrm{Hom}\,(Y, Z^i) \to \mathrm{Hom}\,(Y, \bigoplus_i Z^i)$ to be an isomorphism as well.

This means that the functor $\mathrm{Hom}_{K^b}\,(Y, -)$ commutes with arbitrary direct sums; therefore by induction we find a complex $\overline{Y}$ of finitely generated modules which is isomorphic to Y and bounded in the same range. In particular $\overline{Y}$ does not overlap with X_0.

The desired complex $\overline{X}$ will now be

$$\cdots \to 0 \to X_0 \xrightarrow{d_0'} \overline{Y}_1 \to \cdots \overline{Y}_n \to 0 \to \cdots$$

where the map d_0' is just given as the composition $\lceil X_0 \rfloor \xrightarrow{d_0} Y[1] \cong \overline{Y}[1]$. Clearly $\overline{X}$ inherits its finitely generated components from $\overline{Y}$ and X_0. To see that $\overline{X}$ is isomorphic to X in $K^b\,(\Lambda\text{−Proj})$, complete the diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & X & \longrightarrow & \lceil X_0 \rfloor & \longrightarrow & Y[1] \\ \downarrow\cong & & & & \| & & \downarrow\cong \\ \overline{Y} & \longrightarrow & \overline{X} & \longrightarrow & \lceil X_0 \rfloor & \longrightarrow & \overline{Y}[1] \end{array}$$

to a map of triangles, which then has to give an isomorphism $X \cong \overline{X}$.

This concludes the proof of Theorem 2.1.1.

3. Classification of derived equivalences

3.1. Tilting complexes and Rickard's theorem

Let Λ and Γ be associative and unitary rings. The following definition introduces our main tool for understanding derived equivalences $D^*(\Lambda) \simeq D^*(\Gamma)$.

Definition 3.1.1. *A bounded complex $T \in K^b(\Gamma\text{−proj})$ of finitely generated projective Γ-modules is called a **tilting complex** over Γ, if*

T1 *there are no higher self-extensions of T, i.e. $\mathrm{Hom}_{K^b}(T, T[i]) = 0$ for all $i \neq 0$*

T2 *the class of objects $\mathrm{Add}{-}T$ generates $K^b(\Gamma\text{−proj})$ as a triangulated category, i.e. there are no proper full triangulated subcategories of $K^b(\Gamma\text{−proj})$ containing $\mathrm{Add}{-}T$.*

The easiest and in some sense most important example is the regular module ${}_\Gamma\Gamma$ of the ring Γ, which is a tilting complex when considered as an object $\lceil\Gamma\rfloor$ of $K^b(\Gamma\text{−proj})$. Condition T1 is clearly satisfied and T2 is precisely the following

Lemma 3.1.2. *The class $\mathrm{Add}{-}\lceil\Gamma\rfloor$ (which is nothing but the image of $\Gamma\text{−proj}$ in the derived category) generates $K^b(\Gamma\text{−proj})$ as a triangulated subcategory.*

Proof. Let $\mathcal{V}$ be a full triangulated subcategory of $K^b(\Gamma\text{−proj})$ containing $\mathrm{Add}{-}\lceil\Gamma\rfloor$. Every complex $\lceil P\rfloor_n$ concentrated in one degree is contained in $\mathcal{V}$, since $\mathcal{V}$ contains $\lceil P\rfloor$ and is closed under shifts. To prove that an arbitrary object X of $K^b(\Gamma\text{−proj})$ lies in $\mathcal{V}$, we do induction on the size of the support of X using the standard truncation triangle (see Example B.3.4 in the appendix); we use that if two out of three objects of a triangle lie in a triangulated subcategory, then so does the third. □

Corollary 3.1.3. *As a consequence, T2 is equivalent to*

T2' *The triangulated subcategory of $K^b(\Gamma\text{−proj})$ generated by $\mathrm{Add}{-}T$ contains all the finitely generated projective Γ-modules via $\Gamma\text{−proj} \hookrightarrow K^b(\Gamma\text{−proj})$.* □

Since T1 and T2 are formulated using only the triangulated and additive structure of K^b, both conditions are respected by any equivalence of triangulated categories. Therefore, if we have a triangulated equivalence $K^b(\Gamma\text{−proj}) \xrightarrow{\simeq} K^b(\Lambda\text{−proj})$, then $\lceil\Gamma\rfloor$ gets mapped to a tilting complex T over Λ and there is an isomorphism between $\Gamma^{\mathrm{opp}} = \mathrm{End}_{K^b}(\lceil\Gamma\rfloor)$ and $\mathrm{End}_{K^b}(T)$

The preceding observation has a surprising converse, the proof of which will be the main content of this thesis.

Main Theorem 3.1.4 (Rickard). *Let Λ be any ring and $T \in K^b(\Lambda\text{−proj})$ a tilting complex over Λ. Then the two rings Λ and $\Gamma := \operatorname{End}^{\mathrm{opp}}_{K^b}(T)$ are derived equivalent. More precisely, there is an adjoint equivalence of triangulated categories*

$$F_T \dashv G_T \colon D^-(\Gamma) \simeq K^-(\Gamma\text{−Proj}) \xrightarrow{\simeq} K^-(\Lambda\text{−Proj}) \simeq D^-(\Lambda)$$

which maps $\lceil \Gamma \rfloor$ to T and restricts to triangulated equivalences

- $D^b(\Gamma\text{−Proj}) \xrightarrow{\simeq} D^b(\Lambda\text{−Proj})$,
- $K^b(\Gamma\text{−Proj}) \xrightarrow{\simeq} K^b(\Lambda\text{−Proj})$ *and*
- $K^b(\Gamma\text{−proj}) \xrightarrow{\simeq} K^b(\Lambda\text{−proj})$.

3.2. An easy example

Fix any field k. Let $\Lambda = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$ be the k-algebra of upper triangular 3×3-matrices and let I be the two-sided ideal $\begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; we can view Λ as the path algebra of the quiver $3 \to 2 \to 1$ and I as the ideal generated by the path $\to\to$.
Furthermore, let T be the module $P_1 \oplus P_3 \oplus L_3$, where P_i resp. L_i is the i-th projective resp. i-th simple module. One can either check by hand or use the general theory of quasi-hereditary algebras [KK99, Proposition 2.1] (with the partial ordering $1 < 2 > 3$ on the vertices equipping $P_1 \oplus P_3 \oplus L_3$ with both a standard and a costandard filtration) to see that T has no nontrivial self-extensions. We can view T as an object of $K^b(\Lambda\text{−proj})$ by replacing it by its projective resolution

$$P_2 \xhookrightarrow{\begin{pmatrix}0, 0, \iota\end{pmatrix}} P_1 \oplus P_3 \oplus P_3.$$

Since P_1, P_3 and L_3 are in Add$-T$ and we have the triangle coming from the exact sequence $0 \to P_2 \to P_3 \to L_3 \to 0$, we see that all the indecomposable projective modules are in the triangulated subcategory generated by Add$-T$. Hence T2' is fulfilled and so T2 holds as well.
The only nontrivial homomorphisms between the indecomposable summands of T are $P_1 \hookrightarrow P_3 \twoheadrightarrow L_3$ and the composition is zero. Therefore the (opposite) endomorphism ring of T is nothing but $k(\bullet \to \bullet \to \bullet)/(\to\to)$, which can be identified with Λ/I.
The main theorem tells us now that Λ and Λ/I are derived equivalent. Note, however, that Λ and Λ/I are not Morita equivalent. Indeed, Λ has global dimension one like any path algebra, while it is easy to see that $\operatorname{Ext}^2_{\Lambda/I}(L_3, P_1) \neq 0$.

4. The functor F

In this section we do one half of the construction and proof of the Main Theorem 3.1.4. For some fixed tilting complex T over Λ with opposite endomorphism ring $\Gamma := \mathrm{End}^{\mathrm{opp}}_{K^-}(T)$ we construct the functor $F = F_T\colon D^-(\Gamma) \to D^-(\Lambda)$ and prove that it is triangulated (Theorem 4.4.1) and fully faithful (Theorem 4.5.1).

4.1. Equivalence of $D^-(\Gamma)$ and $K^-(\mathrm{Sum}{-}T)$

First of all we observe that we can replace $D^-(\Gamma) \simeq K^-(\Gamma{-}\mathrm{Proj})$ by the equivalent category $K^-(\Gamma{-}\mathrm{Free})$. This is true since every right-bounded complex

$$\cdots \to X_{n-2} \to X_{n-1} \to X_n \to 0 \to \cdots$$

of projective modules can be made free by adding the contractible complex

$$\cdots \to X'_{n-2} \oplus X'_{n-1} \to X'_{n-1} \oplus X'_n \to X'_n \to 0 \to \cdots,$$

where X'_n is chosen such that $X'_n \oplus X_n$ is free and X'_i for $i < 0$ is recursively chosen such that $X'_i \oplus \left(X'_{i+1} \oplus X_i\right)$ is free.

Since the functor F is supposed to map $\lceil \Gamma \rfloor$ to T, it stands to reason that there should be a close relationship between sums of copies of Γ and sums of copies of T. The following lemma tells us that this is true in the strongest possible sense: there is a canonical identification between $\mathrm{Sum}{-}T$ and $\Gamma{-}\mathrm{Free}$.

Lemma 4.1.1. *The functor*

$$\mathbb{T} = \mathrm{Hom}_{K^b(\Lambda-\mathrm{Proj})}(T, -) : K^b(\Lambda{-}\mathrm{Proj}) \longrightarrow \Gamma{-}\mathrm{Mod}$$

restricts to an equivalence of categories $\mathrm{Sum}{-}T \simeq \Gamma{-}\mathrm{Free}$.

Proof. Theorem 2.1.1 (iii) tells us that $\mathrm{Hom}(T, -)$ commutes with arbitrary direct sums, since T is a bounded complex with finitely generated entries. Therefore for any index set I, the sum $T^{(I)}$ is mapped to $\mathrm{Hom}(T, T)^{(I)}$, which as a Γ-module is nothing but the free module $\Gamma^{(I)}$. This shows that $\mathbb{T}$ induces a dense functor $\mathrm{Sum}{-}T \to \Gamma{-}\mathrm{Free}$.
Moreover, the map induced by the functor on Hom-sets

$$\begin{array}{rcl} \Gamma^{\mathrm{opp}} = \mathrm{Hom}_{K^b}(T, T) & \longrightarrow & \mathrm{Hom}_\Gamma(\Gamma, \Gamma) = \mathrm{End}_\Gamma(\Gamma) \\ f & \mapsto & f \circ - \end{array}$$

is just the opposite of the canonical isomorphism $\Gamma \xrightarrow{\cong} \mathrm{End}_\Gamma^{\mathrm{opp}}(\Gamma)$ which sends γ to the right-multiplication with γ.
This immediately proves that $\mathbb{T}$ is fully faithful when restricted to Sum$-T$, since it induces an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{K^b}\left(\bigoplus_{i\in I} T, \bigoplus_{j\in J} T\right) & \xrightarrow{\cong} & \prod_{i\in I}\bigoplus_{j\in J} \mathrm{Hom}_{K^b}(T,T) \\ & & \downarrow{\scriptstyle \mathbb{T}}\,{\scriptstyle\cong} \\ \mathrm{Hom}_\Gamma\left(\bigoplus_{i\in I} \Gamma, \bigoplus_{j\in J} \Gamma\right) & \xleftarrow{\cong} & \prod_{i\in I}\bigoplus_{j\in J} \mathrm{Hom}_\Gamma(\Gamma,\Gamma) \end{array}$$

for any two index sets I and J. Hence $\mathbb{T}$ restricts to the desired equivalence. □

Passing to the homotopy category of complexes, we get an equivalence of triangulated categories

$$K^-(\mathbb{T}) : K^-(\mathrm{Sum}-T) \xrightarrow{\simeq} K^-(\Gamma-\mathrm{Free}),$$

which maps $\lceil T \rfloor$ to $\lceil \Gamma \rfloor$. We will identify the two categories and mostly work with the first one, as it is closer to what we want. Indeed, we see that the objects of $K^-(\mathrm{Sum}-T)$ are complexes of complexes of projective Λ-modules (with some homotopies involved), which isn't too far from the category $K^-(\Lambda-\mathrm{Proj})$ we are aiming for.

4.2. Λ-grids and augmented bicomplexes

In order to increase conceptual clarity, we introduce a new name for an already well known object.

Definition 4.2.1. *A Λ-**grid** $X = X^{*,*}$ is a $\mathbb{Z}\times\mathbb{Z}$-indexed family $(X^{i,j})_{i,j\in\mathbb{Z}}$ of Λ-modules. For two Λ-grids X and Y and any $a,b\in\mathbb{Z}$, a **grid map** (or **morphism of grids**) $\varphi = \varphi^{*,*} : X \to Y$ of **bidegree** (a,b) is a collection of Λ-homomorphisms $\varphi^{i,j} : X^{i,j} \to Y^{i+a,j+b}$. There is an obvious way to compose grid maps which is additive on the bidegrees.*
*We call a Λ-grid X **projective**, if all the components $X^{i,j}$ for $i,j\in\mathbb{Z}$ are projective Λ-modules.*
*A Λ-grid X is called **horizontally** (resp. **vertically**) **bounded**, if there are bounds $l\le r$ such that $X^{i,*}$ (resp. $X^{*,i}$) is zero unless $l\le i\le r$.*

Example 4.2.2. *A bicomplex is nothing but a Λ-grid together with commuting endomorphisms c and δ of bidegree $(1,0)$ and $(0,1)$ respectively, such that $cc=0$ and $\delta\delta=0$.*

Every object X of $K^-(\text{Sum}-T)$ has an underlying projective Λ-grid $X^{*,*}$ given by $X^{i,j} = (X_j)_i$. This expression makes sense, because every X_j is an object of Sum$-T$, hence it is a complex of projective Λ-modules with components $(X_j)_i$. The Λ-grid $X^{*,*}$ is horizontally bounded as T is a bounded complex.
Moreover, we can convert the differentials $(c_j)_i : (X_j)_i \to (X_j)_{i+1}$ of the various complexes $X_j \in \text{Sum}-T$ into one grid map $c\colon X^{*,*} \to X^{*,*}$ of bidegree $(1,0)$. This c will be called the **horizontal differential** of X (or of $X^{*,*}$).
One could hope to extract from the homological differentials of X (i.e. the maps $\hat{\delta}_j\colon X_j \to X_{j+1}$) a second grid map δ of bidegree $(0,1)$, turning $X^{*,*}$ into a bicomplex. If this were possible, we could simply take the total complex of $X^{*,*}$ and get an object in $K^-(\Lambda-\text{Proj})$. We would then have found an assignment on objects

$$\text{Ob}\, K^-(\text{Sum}-T) \longrightarrow \text{Ob}\, K^-(\Lambda-\text{Proj}),$$

which we could hope to extend to the desired functor F.
Sadly, this does not work in general. Note, in fact, that for any $j \in \mathbb{Z}$ the map $\hat{\delta}_j\colon X_j \to X_{j+1}$, which is an arrow of the category $\text{Sum}-T \subset K^-(\Lambda-\text{Proj})$, is not an actual map of complexes, but a homotopy class of chain maps. Therefore we have to choose some representatives δ_j of $\hat{\delta}_j$. The problem with this is that $\delta_{j+1}\delta_j$ need not be zero in general. The fact that X is a complex, i.e. $\hat{\delta}_{j+1}\hat{\delta}_j = 0$ as arrows in Sum$-T$, only tells us that $\delta_{j+1}\delta_j$ is *homotopic* to zero as a map of complexes. Hence if we put the δ_j's together to a grid map $\delta\colon X^{*,*} \to X^{*,*}$ of bidegree $(0,1)$, there is no reason why $\delta\delta$ should vanish.

Remark 4.2.1. When we speak of a **vertical differential** of X (or $X^{*,*}$) we mean a *choice* $\delta\colon X^{*,*} \to X^{*,*}$ of representatives as in the previous paragraph. This will allow us to identify each object X of $K^-(\text{Sum}-T)$ with the triple $(X^{*,*}, c, \delta)$ which by slight abuse of terminology we also call its **underlying Λ-grid**.
Similarly each chain map $\hat{\alpha} \in \text{Hom}_{C^-}(X, Y)$ will be identified with a grid map $\alpha\colon X^{*,*} \to Y^{*,*}$ of bidegree $(0,0)$ by choosing representatives α_j of the components $\hat{\alpha}_j\colon X_j \to Y_j$.

The idea of turning an object of $K^-(\text{Sum}-T)$ into a bicomplex and then taking total complexes is not completely misguided, though. We just have to use a more sophisticated version of the concepts 'bicomplex' and 'total complex'.

Definition 4.2.3. *We define the category $\mathfrak{B}(\Lambda)$ of* ***augmented bicomplexes*** *as follows:*

(1) Objects are horizontally bounded projective Λ-grids $X = X^{*,*}$ together with a family of grid endomorphisms $\left(d_i = d_i^{*,*} = d_i^X \colon X \to X\right)_{i\in\mathbb{N}}$ of bidegree $(1-i, i)$

such that for each natural number k we have

$$\sum_{i+j=k} d_i d_j = 0. \tag{4.1}$$

(2) *A morphism $\alpha\colon X \to Y$ is a family of grid maps $(\alpha_i = \alpha_i^{*,*}\colon X \to Y)_{i\in\mathbb{N}}$ of bidegree $(-i,i)$ such that for each natural number k we have*

$$\sum_{i+j=k} \alpha_i d_j^X = \sum_{i+j=k} d_j^Y \alpha_i. \tag{4.2}$$

(3) *The identity morphism $X \to X$ is given by $\mathbb{1}_0 = \mathbb{1}_X$ and $\mathbb{1}_k = 0$ for all higher k. The composition $\alpha\beta\colon X \to Z$ of two morphisms $\alpha\colon Y \to Z$ and $\beta\colon X \to Y$ has components*

$$(\alpha\beta)_k \coloneqq \sum_{i+j=k} \alpha_i \beta_j.$$

Moreover, we have a canonical functor $\operatorname{tot}\colon \mathfrak{B}(\Lambda) \to C^-(\Lambda\text{−Proj})$ *which just sums up all the modules, differentials and morphisms diagonally, i.e it is*

- *on objects:* $\operatorname{tot}(X)_n = \bigoplus_{i+j=n} X^{i,j}$ *with differential*

$$d_n^{\operatorname{tot}(X)} = \sum_{i+j=n} \sum_{k\in\mathbb{N}} \left(d_k^{i,j}\colon X^{i,j} \to X^{i+1-k,j+k}\right),$$

- *on morphisms:*

$$\operatorname{tot}(\alpha\colon X \to Y)_n = \sum_{i+j=n} \sum_{k\in\mathbb{N}} \left(\alpha_k^{i,j}\colon X^{i,j} \to X^{i-k,i+k}\right).$$

Remark 4.2.2. It is immediate that the structure given in (1)-(3) defines a category. The fact that X is horizontally bounded guarantees that all the sums $\bigoplus_{i+j=n} X^{i,j}$ are finite; we can therefore sum up morphisms on different components without worrying about finiteness conditions. The functor tot is well-defined since $\operatorname{tot}(X)_n$ is projective and

$$d_{n+1}^{\operatorname{tot}(X)} d_n^{\operatorname{tot}(X)} = \sum_{i+j=n} \sum_{k\in\mathbb{N}} \left(\sum_{r+l=k} d_r^{i+1-r,j+r} d_l^{i,j} \right) = 0$$

by axiom (4.1) (so $\operatorname{tot}(X)$ is a complex); similarly condition (4.2) on α ensures that $\operatorname{tot}(\alpha)$ commutes with d^{tot} (so $\operatorname{tot}(\alpha)$ is a morphism of complexes).

Example 4.2.4. *There is a canonical way to view a bicomplex* (X, c, δ) *as an augmented bicomplex. Indeed, we can take* $d_0 = c$ *and* $d_1^{i,j} = (-1)^{i+j}\,\delta^{i,j}$ *as well as* $d_k = 0$ *for all higher* $k > 1$.
Moreover, the usual total complex of the bicomplex (X, c, δ) *agrees with* $\mathrm{tot}\,(X^{*,*}, d)$ *defined in Definition 4.2.3.*

Definition 4.2.5. *A* ***homotopy*** $h\colon (\alpha\colon X \to Y) \simeq (\beta\colon X \to Y)$ *between two parallel morphisms of augmented bicomplexes is a family of maps* $(h_i\colon X \to Y)_{i\in\mathbb{N}}$ *of bidegree* $(-1-i, i)$ *such that for each natural number* k *we have*

$$\alpha_k - \beta_k = \sum_{i+j=k} \left(d_i^Y h_j + h_j d_i^X\right).$$

We obtain the homotopy category $\mathfrak{B}/{\simeq}$ *of augmented bicomplexes by identifying homotopic maps.*

It is easy to see that each homotopy $h\colon \alpha \simeq \beta$ in $\mathfrak{B}\,(\Lambda)$ gives rise to a homotopy $\mathrm{tot}\,(h):\ \mathrm{tot}\,(\alpha) \simeq \mathrm{tot}\,(\beta)$ by $\mathrm{tot}\,(h\colon \alpha \simeq \beta)_n = \sum_{i+j=n} \sum_{k\in\mathbb{N}} \left(h_k^{i,j}\colon X^{i,j} \to Y^{i-1-k,j+k}\right)$. This means that we get a functor $\mathfrak{B}\,(\Lambda)/{\simeq} \to K^-\,(\Lambda{-}\mathrm{Proj})$ which we will also denote by tot.

4.3. Embedding of $C^-\,(\mathrm{Sum}{-}T)$ into $\mathfrak{B}\,(\Lambda)$

Proposition 4.3.1. *(i) For each object* X *of* $C^-\,(\mathrm{Sum}{-}T)$*, we can extend the underlying* Λ*-grid* $(X^{*,*}, c, \delta)$ *to an augmented bicomplex* $\vartheta(X) = (X^{*,*}, d)$ *with* $d_0 = c$ *and* $d_1 = \pm\delta$.

(ii) For each morphism $\hat{\alpha}\colon X \to Y$*, we can pick some underlying grid map* α *and extend it to a morphism of augmented bicomplexes* $\vartheta\,(\hat{\alpha}) : \vartheta\,(X) \to \vartheta\,(Y)$ *with* $\vartheta\,(\hat{\alpha})_0 = \alpha$.

Moreover, the choices we make differ only by a homotopy in $\mathfrak{B}\,(\Lambda)$ *and therefore* ϑ *induces a functor*

$$C^-\,(\mathrm{Sum}{-}T) \longrightarrow \mathfrak{B}\,(\Lambda)/{\simeq}.$$

We will prove Proposition 4.3.1 by inductively constructing the data necessary to define objects and morphisms in $\mathfrak{B}\,(\Lambda)$ using the following

Lemma 4.3.2. *Let* X*,* Y *be objects in* $C^-\,(\mathrm{Sum}{-}T)$ *viewed as* Λ*-grids. Let* c^X*,* c^Y *be the horizontal differentials of* X *and* Y *respectively. Let* $\beta\colon X \to Y$ *be a grid map of bidegree* (p, q) *with* $p \neq 0$ *commuting with* c*, i.e. satisfying* $\beta c^X = c^Y \beta$*. Then we can find a grid map* $t\colon X \to Y$ *of bidegree* $(p-1, q)$ *with* $\beta = c^Y t + t c^X$.

Proof. We fix the second degree j and consider the map $\beta_j =: X^{*,j} \to Y^{*+p,j+q}$. The condition on β says precisely that all the β_j's are morphisms of complexes. Both $X^{*,j}$ and $Y^{*,j+q}$ are sums of copies of T and (up to the sign of the differential) $Y^{*+p,j+q}$ is just $Y^{*,j+q}[p]$, i.e a nontrivial shift $[p]$ of a sum of copies of T. Thus, since T is a tilting complex, β_j has to be nullhomotopic, i.e. we find $t_j \colon X^{*,j} \to Y^{*+p-1,j+q}$ with $\beta_j = c^Y_{j+q} t_j + t_j c^X_j$. Putting the t_j together for all $j \in \mathbb{Z}$ gives the desired t. $\square$

Proof of Proposition 4.3.1. Let X be an object in $C^-(\mathrm{Sum}{-}T)$, we need to construct the differentials d_i of $\vartheta(X)$ starting from $d_0 = c$ and $d_1 = \pm\delta$. We see that $d_0 d_0 = 0$ and $d_0 d_1 + d_1 d_0 = \pm c\delta \mp \delta c = 0$, since δ is a morphism of complexes, i.e. commutes with c.
Oberve that $\delta\delta$ is homotopic to zero, so there is a homotopy d_2 with $\delta\delta = cd_2 + d_2 c$. This d_2 will then be a grid map of bidegree $(1, -2)$ and will satisfy

$$d_0 d_2 + d_1 d_1 + d_2 d_0 = cd_2 - \delta\delta + d_2 c = 0,$$

which is precisely the required identity (4.1).
We assume now by induction for $n > 2$ that we have already constructed d_i for $i = 0, \ldots, n-1$ satisfying $\sum_{i+j=k} d_i d_j = 0$ for all $k = 0, \ldots, n-1$ and we consider the map

$$\beta := -\sum_{\substack{i+j=n\\ i,j\neq 0}} d_i d_j,$$

which has bidegree $(2-n, n)$.
Using the induction hypothesis, it is an easy calculation that β commutes with c $(= d_0)$:

$$\begin{aligned} d_0\beta &= -\sum_{\substack{i+j=n\\ i,j\neq 0}} (d_0 d_i)\, d_j = \sum_{\substack{i+j=n\\ i,j\neq 0}} \left(\sum_{\substack{k+l=i\\ k\neq 0}} d_k d_l \right) d_j \\ &= \sum_{\substack{k+l+j=n\\ k,j\neq 0}} d_k d_l d_j = \ldots = \beta d_0. \end{aligned}$$

Since $2-n \neq 0$ we can use Lemma 4.3.2 and find $d_n := t$ of bidegree $(1-n, n)$ with $\beta = d_0 d_n + d_n d_0$, i.e. $\sum_{i+j=n} d_i d_j = 0$.

Now we need to do the same with morphisms $\hat{\alpha} \colon X \to Y$, with α being some underlying grid map of $\hat{\alpha}$. We have $\alpha c = c\alpha$, since each $\alpha_j \colon X_j \to Y_j$ is a morphism of complexes and therefore we can define $\overline{\alpha}_0 := \alpha$ to fulfil (4.2).
Since $\hat{\alpha}$ is a morphism of complexes with entries in $\mathrm{Sum}{-}T$, it commutes with

$\hat{\delta}$. This means that α commutes with δ up to homotopy, i.e. we find $t\colon X \to Y$ of bidegree $(-1,1)$ with $c^Y t + t c^X = \alpha\delta^X - \delta^Y\alpha$. Setting $\overline{\alpha}_1 \coloneqq \pm t$, i.e. $\overline{\alpha}_1^{i,j} = (-1)^{i+j}\, t^{i,j}$ gives

$$\overline{\alpha}_0 d_1^X + \overline{\alpha}_1 d_0^X = \pm\left(\alpha\delta^X - tc^X\right) = \pm\left(c^Y t + \delta^Y\alpha\right) = d_0^Y\overline{\alpha}_1 + d_1^Y\overline{\alpha}_0,$$

which is precisely (4.2).

If for $n > 1$ the map $\overline{\alpha}_i$ is already defined for $i = 0, \ldots, n-1$, satisfying $\sum_{i+j=k} \overline{\alpha}_i d_j = \sum_{i+j=k} d_i\overline{\alpha}_j$ for all $k = 0, \ldots, n-1$ (note that we are dropping the superscripts X and Y for the d's), then we can define

$$\gamma = \sum_{\substack{i+j=n \\ j\neq 0}} \overline{\alpha}_i d_j - d_j\overline{\alpha}_i$$

which has bidegree $(1-n, n)$. We check that γ commutes with the differential d_0 up to a sign:

$$\begin{aligned}
d_0\gamma &= \sum_{\substack{i+j=n \\ j\neq 0}} \left(d_0\overline{\alpha}_i d_j - (d_0 d_j)\,\overline{\alpha}_i\right) \\
&= \sum_{\substack{i+j=n \\ j\neq 0}} \left(d_0\overline{\alpha}_i d_j + \left(d_j d_0 + \sum_{\substack{k+l=j \\ k,l\neq 0}} d_k d_l\right)\overline{\alpha}_i\right) \\
&= \left(\sum_{\substack{i+j=n \\ j\neq 0}} d_0\overline{\alpha}_i d_j + d_j\left(d_0\overline{\alpha}_i\right)\right) + \left(\sum_{\substack{k+l+i=n \\ k,l\neq 0}} d_k d_l\overline{\alpha}_i\right) \\
&= \left(\sum_{\substack{i+j=n \\ j\neq 0}} d_0\overline{\alpha}_i d_j + d_j\left(\overline{\alpha}_i d_0 + \sum_{\substack{k+l=i \\ l\neq 0}} \left(\overline{\alpha}_k d_l - d_l\overline{\alpha}_k\right)\right)\right) + \left(\sum_{\substack{j+l+k=n \\ j,l\neq 0}} d_j d_l\overline{\alpha}_k\right) \\
&= \left(\sum_{\substack{i+j=n \\ j\neq 0}} d_0\overline{\alpha}_i d_j + d_j\overline{\alpha}_i d_0\right) + \left(\sum_{\substack{j+k+l=n \\ j,l\neq 0}} d_j\overline{\alpha}_k d_l\right) = \cdots = -\gamma d_0.
\end{aligned}$$

Therefore applying Lemma 4.3.2 to $\beta \coloneqq \pm\gamma$ (i.e. $\beta^{i,j} = (-1)^{i+j}\,\gamma^{i,j}$) gives us some t with $\pm\gamma = d_0 t + t d_0$. We set $\overline{\alpha}_n \coloneqq \pm t$ and see that it satisfies $\sum_{i+j=n} \overline{\alpha}_i d_j = \sum_{i+j=n} d_j\overline{\alpha}_i$ as desired.

Finally we can take $\vartheta\left(\hat{\alpha}\right)$ to be $\left(\overline{\alpha}_i\right)_{i\in\mathbb{N}}$.

Finally we need to check that if we apply this construction to two maps α and α' whose components α_j and α'_j are homotopic (as maps between complexes of Λ-modules), then we obtain homotopic results. From this it follows that the assignment ϑ is functorial up to homotopy.
Without loss of generality, we may assume that α' is zero. Choose homotopies $h_j\colon \alpha_j \simeq 0$ and collect them to a map h which then satisfies $\overline{\alpha}_0 = \alpha = h_0 d_0 + d_0 h_0$. Assume by induction that we have defined h_i already for $i = 0, \dots, n-1$ such that $\overline{\alpha}_k = \sum_{i+j=k} (d_i h_j + h_j d_i)$ for all $k = 0, \dots, n-1$.
Then we set $\beta = \overline{\alpha}_n - \sum_{\substack{i+j=n \\ i\neq 0}} (d_i h_j + h_j d_i)$ and after checking that β commutes with d_0 we obtain the desired $h_n \coloneqq t$ using Lemma 4.3.2. □

From now on, we will commit a slight abuse of notation and think of $C^-(\text{Sum}{-}T)$ as embedded into $\mathfrak{B}(\Lambda)$ via ϑ, even though technically ϑ is only an embedding up to homotopy. We will therefore not distinguish between a complex with entries in Sum$-T$ and the associated augmented bicomplex; we will identify maps between such complexes with the morphisms of augmented bicomplexes given by Proposition 4.3.1.

4.4. Construction of the triangulated functor F

Thanks to Proposition 4.3.1 we can now define the functor $\hat{F}$ as the composition

$$C^-(\text{Sum}{-}T) \xrightarrow{\vartheta} \mathfrak{B}(\Lambda)/{\simeq} \xrightarrow{\text{tot}} K^-(\Lambda{-}\text{Proj}).$$

We will then get the desired functor $F\colon K^-(\text{Sum}{-}T) \to K^-(\Lambda{-}\text{Proj})$ once we have proved that $\hat{F}$ factors through the homotopy category.

Remark 4.4.1. If the vertical differentials of an object X in $C^-(\text{Sum}{-}T)$ square to zero on the nose and not only up to homotopy, then the underlying Λ-grid of X will be a bicomplex so that Example 4.2.4 applies.
A particular case is if we start with a sum U of copies of T, viewed as a complex $\lceil U \rfloor_m$ concentrated in some degree m, and apply $\hat{F}$ to it. In this case we get back (up to the sign of the differentials) a shifted version $U[-m]$ of what we had, because all we are doing is taking the total complex of a bicomplex which has only one nonzero row. In other words, the composition

$$\text{Sum}{-}T \xhookrightarrow{\lceil - \rfloor_m} K^-(\text{Sum}{-}T) \xrightarrow{F} K^-(\Lambda{-}\text{Proj})$$

is (up to the sign of the differentials) just the shift functor $[-m]$ restricted to the full subcategory Sum$-T$ of $K^-(\Lambda-\mathrm{Proj})$.

Theorem 4.4.1. *The functor*

$$\hat{F}\colon C^-(\mathrm{Sum}-T)\longrightarrow K^-(\Lambda-\mathrm{Proj})$$

factors through $K^-(\mathrm{Sum}-T)$ *and the resulting functor*

$$F\colon K^-(\mathrm{Sum}-T)\longrightarrow K^-(\Lambda-\mathrm{Proj})$$

is triangulated, i.e. commutes with the shift functor and maps triangles to triangles.

In order to prove this theorem, we need the following

Lemma 4.4.2. *Let $\mathcal{A}$ be an additive category and let $C^*(\mathcal{A})$ be a category of complexes over $\mathcal{A}$, where $* \in \{\emptyset,+,-,b\}$. Let $\mathcal{T}$ be a triangulated category and*

$$\hat{F}\colon C^*(\mathcal{A})\longrightarrow \mathcal{T}$$

an additive functor which is "triangulated", in the following sense:
$\hat{F}$ commutes with the suspension functors and each "triangle" in $C^*(\mathcal{A})$ of the form

$$X\xrightarrow{\alpha} Y\to C(\alpha)\to X[1],$$

where $C(\alpha)$ is the mapping cone of α, gets mapped by $\hat{F}$ to an actual triangle in $\mathcal{T}$

$$\hat{F}X\xrightarrow{\hat{F}\alpha}\hat{F}Y\to C\left(\hat{F}\alpha\right)\rightsquigarrow.$$

Then $\hat{F}$ factors through $K^*(\mathcal{A})$ and the resulting functor is triangulated.

Proof. For any object X in $C^*(\mathcal{A})$ consider the "triangle"

$$X\xrightarrow{\mathbb{1}_X} X\to C(\mathbb{1}_X)\to X[1].$$

Since $\hat{F}$ maps this "triangle" to a triangle and $\mathbb{1}_X$ to $\mathbb{1}_{\hat{F}X}$, it must map $C(\mathbb{1}_X)$ to zero. By Proposition B.2.2 (ii) an object X is contractible if and only if the identity on X factors through the canonical map $X\to C(\mathbb{1}_X)$. This means that if X if contractible, the identity on $\hat{F}X$ factors through $\hat{F}C(\mathbb{1}_X)$ which is zero. Therefore $\hat{F}X$ has to be zero. We have shown in this way that $\hat{F}$ kills all contractible complexes.

If $\alpha\colon X\to Y$ is a homotopy equivalence, then Proposition B.2.2 (iii) tells us that its cone $C(\alpha)$ is contractible and is thus killed by $\hat{F}$. Since we have a triangle $\hat{F}X\xrightarrow{\hat{F}\alpha}\hat{F}Y\to\hat{F}(C(\alpha))\rightsquigarrow$ in $\mathcal{T}$, this means that $\hat{F}\alpha$ is an isomorphism. This proves that $\hat{F}$ sends all homotopy equivalences to isomorphisms.

Now observe that Proposition B.2.2 (vi) tells us (in the notation there) that p is a homotopy equivalence and therefore $\hat{F}p$ is an isomorphism. Since pi_0 and pi_1 are equal, we have $\hat{F}p\hat{F}i_0 = \hat{F}p\hat{F}i_1$; hence $\hat{F}i_0 = \hat{F}i_1$.
Finally, let $g_0, g_1 \colon X \to Y$ be two homotopic maps. Using Proposition B.2.2, (v) we find a map $h \colon M(\mathbb{1}_X) \to Y$ with $g_0 = hi_0$ and $g_1 = hi_1$. We have then equalities $\hat{F}g_0 = \hat{F}h\hat{F}i_0 = \hat{F}h\hat{F}i_1 = \hat{F}g_1$.
This shows that $\hat{F}$ is well-defined on homotopy classes, hence it factors through the homotopy category $K^*(\mathcal{A})$.

It is clear that the resulting functor is triangulated, since all the conditions for being triangulated are part of the assumptions on $\hat{F}$. $\square$

Proof of Theorem 4.4.1. By Lemma 4.4.2 it is enough to show that for each morphism $\alpha \colon X \to Y$ in $C^-(\mathrm{Sum}-T)$ there is the following commutative diagram in which the vertical maps are isomorphisms and η^X and η^Y depend only on X and Y respectively (but not on α):

$$\begin{array}{ccccccccc} \hat{F}X & \xrightarrow{\hat{F}\alpha} & \hat{F}Y & \longrightarrow & \hat{F}(C(\alpha)) & \longrightarrow & \hat{F}(X[1]) & \xrightarrow{\hat{F}(\alpha[1])} & \hat{F}(Y[1]) \\ \| & & \| & & \downarrow\cong & & \eta^X\downarrow\cong & & \eta^Y\downarrow\cong \\ \hat{F}X & \xrightarrow{\hat{F}\alpha} & \hat{F}Y & \longrightarrow & C\left(\hat{F}\alpha\right) & \longrightarrow & \left(\hat{F}X\right)[1] & \xrightarrow{(\hat{F}\alpha)[1]} & \left(\hat{F}Y\right)[1] \end{array}.$$

Note that the commutativity of the rightmost square tells us precisely that collecting the η^X together gives an isomorphism $\eta \colon \hat{F} \circ [1] \xrightarrow{\cong} [1] \circ \hat{F}$ of functors.

First of all, let us fix some notation. Let γ and δ be the vertical differentials of Y and X, respectively and let $(c_k)_{k\in\mathbb{N}}$ resp. $(d)_{k\in\mathbb{N}}$ be the augmented differentials of $Y^{*,*}$ resp. $X^{*,*}$. Moreover, let $\hat{d}$ denote the (augmented) differential of the shifted complex $X[1]$.

We now analyze the interaction of $\hat{F}$ with [1]. Remember, that when applying [1] to a complex, we not only change the indexing by one, but we also change all the signs of the differentials. Therefore we obtain the following identity of maps $X^{i,j} \to X^{i,j+1}$

$$\hat{d}_1^{i,j} = (-1)^{i+j}\delta[1]^{i,j} = -(-1)^{i+j}\delta^{i,j+1} = (-1)^{i+(j+1)}\delta^{i,j+1} = d_1^{i,j+1}.$$

Moreover, the horizontal differential of X does not change when shifting vertically,

so $\hat{d}_0 = d_0$. We conclude that we might as well take $\hat{d}$ to be just d itself (shifted by one, of course). Basically, this just means that the annoying sign-change of the shift functor goes away when we pass from C^- to the augmented category B.
Taking total complexes certainly commutes with shifting the complex, so we get that $\hat{F}(X[1])$ is the same as $\hat{F}X$ shifted by one but without the sign change which appears in $\left(\hat{F}X\right)[1]$. Now we can define the desired isomorphism η_n^X as

$$\hat{F}(X[1])_n \xrightarrow{(-1)^{n+1}\mathbb{1}} \left(\left(\hat{F}X\right)[1]\right)_n,$$

which is clearly natural in X.

Next, we need to find out how $Z := C(\alpha)$ looks like, when we interpret it as an augmented bicomplex $\left(Z^{*,*}, (e_k)_{k\in\mathbb{N}}\right)$ in $\mathfrak{B}(\Lambda)$. Note that on the level of Λ-grids we have $Z^{*,*} = Y^{*,*} \oplus X^{*,*+1}$. We claim that the following is a possible choice of differentials:

$$e_k^{i,j} := \begin{pmatrix} c_k^{i,j} & (-1)^{i+j}\alpha_{k-1} \\ 0 & d_k^{i,j+1} \end{pmatrix} : Y^{i,j} \oplus X^{i,j+1} \to Y^{i+1-k,j+k} \oplus X^{i+1-k,j+1+k},$$

where $(\alpha_k)_{k\in\mathbb{N}}$ is the augmented version of α (we use the convention $\alpha_{-1} = 0$).
First we need to check that e_0 and e_1 agree with the horizontal and up to a sign with the vertical differentials of $C(\alpha)$. The horizontal differential of $C(\alpha)$ is just given by the individual horizontal differentials of X and Y, i.e. it is exactly $\begin{pmatrix} c_0 & 0 \\ 0 & d_0 \end{pmatrix} = e_0$. The vertical differential of $C(\alpha)$ is $\begin{pmatrix} \gamma & \alpha \\ 0 & -\delta \end{pmatrix}$, hence it is equal to $e_1 = \begin{pmatrix} c_1 & \pm\alpha_0 \\ 0 & d_1^{*,*+1} \end{pmatrix} = \begin{pmatrix} \pm\gamma & \pm\alpha \\ 0 & \mp\delta \end{pmatrix}$ up to a sign.
Finally, observe that for all natural numbers n we have

$$\sum_{k+l=n} e_k e_l = \begin{pmatrix} \sum\limits_{k+l=n} c_k c_l & \sum\limits_{k+l=n-1} \pm c_k\alpha_l \mp \alpha_k d_l \\ 0 & \sum\limits_{k+l=n} d_k d_l \end{pmatrix} = 0$$

by the conditions (4.1) and (4.2) on c, d and α; this shows that $(e_k)_{k\in\mathbb{N}}$ is a valid augmentation of the differentials e_0 and e_1.

We take total complexes of $X^{*,*} \xrightarrow{\alpha} Y^{*,*} \to Z^{*,*}$ and get the following chain maps:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \bigoplus_{n} X^{i,j} & \xrightarrow{\sum d} & \bigoplus_{n+1} X^{i,j} & \longrightarrow \cdots \\
& \big\downarrow \sum \alpha & & \big\downarrow \sum \alpha & \\
\cdots \longrightarrow & \bigoplus_{n} Y^{i,j} & \xrightarrow{\sum c} & \bigoplus_{n+1} Y^{i,j} & \longrightarrow \cdots \\
& \big\downarrow \oplus \binom{\mathbb{1}}{0} & & \big\downarrow \oplus \binom{\mathbb{1}}{0} & \\
\cdots \longrightarrow & \bigoplus_{n} Y^{i,j} \oplus X^{i,j+1} & \xrightarrow{\sum e} & \bigoplus_{n+1} Y^{i,j} \oplus X^{i,j+1} & \longrightarrow \cdots \\
& \Vert & & \Vert & \\
\cdots \longrightarrow & \bigoplus_{n} Y^{i,j} \oplus \bigoplus_{n+1} X^{i,j} & \xrightarrow{\begin{pmatrix} \sum c & (-1)^n \sum \alpha \\ 0 & \sum d \end{pmatrix}} & \bigoplus_{n+1} Y^{i,j} \oplus \bigoplus_{n+2} X^{i,j} & \longrightarrow \cdots
\end{array}$$

On the other hand we can form the mapping cone of the map $\hat{F}\alpha = \sum \alpha$ and obtain a similar diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \bigoplus_{n} X^{i,j} & \xrightarrow{\sum d} & \bigoplus_{n+1} X^{i,j} & \longrightarrow \cdots \\
& \big\downarrow \sum \alpha & & \big\downarrow \sum \alpha & \\
\cdots \longrightarrow & \bigoplus_{n} Y^{i,j} & \xrightarrow{\sum c} & \bigoplus_{n+1} Y^{i,j} & \longrightarrow \cdots \\
& \big\downarrow \oplus \binom{\mathbb{1}}{0} & & \big\downarrow \oplus \binom{\mathbb{1}}{0} & \\
\cdots \longrightarrow & \bigoplus_{n} Y^{i,j} \oplus \bigoplus_{n+1} X^{i,j} & \xrightarrow{\begin{pmatrix} \sum c & \sum \alpha \\ 0 & -\sum d \end{pmatrix}} & \bigoplus_{n+1} Y^{i,j} \oplus \bigoplus_{n+2} X^{i,j} & \longrightarrow \cdots
\end{array}$$

Finally, we can identify the two diagrams via the following isomorphism:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \bigoplus_{n} Y^{i,j} \oplus \bigoplus_{n+1} X^{i,j} & \xrightarrow{\begin{pmatrix} \sum c & (-1)^n \sum \alpha \\ 0 & \sum d \end{pmatrix}} & \bigoplus_{n+1} Y^{i,j} \oplus \bigoplus_{n+2} X^{i,j} & \longrightarrow \cdots \\
& \mathbb{1}_Y \oplus (-1)^{n+1} \mathbb{1}_X \big\downarrow \cong & & \cong \big\downarrow \mathbb{1}_Y \oplus (-1)^{n+2} \mathbb{1}_X & \\
\cdots \longrightarrow & \bigoplus_{n} Y^{i,j} \oplus \bigoplus_{n+1} X^{i,j} & \xrightarrow{\begin{pmatrix} \sum c & \sum \alpha \\ 0 & -\sum d \end{pmatrix}} & \bigoplus_{n+1} Y^{i,j} \oplus \bigoplus_{n+2} X^{i,j} & \longrightarrow \cdots
\end{array}$$

It is therefore enough to show that the square

$$\begin{array}{ccc}
\hat{F}\left(C\left(\alpha\right)\right) & \longrightarrow & \hat{F}\left(X[1]\right) \\
\big\downarrow \cong & & \big\downarrow \cong \\
C\left(\hat{F}\alpha\right) & \longrightarrow & \left(\hat{F}X\right)[1]
\end{array}$$

commutes in every degree n. This is true since the horizontal maps are just projections onto the X-component, while both vertical isomorphisms are just $(-1)^{n+1}$ on that component. □

4.5. F is a full embedding

We are finally ready to state and prove the main result of Section 4.

Theorem 4.5.1. *The triangulated functor*

$$F\colon K^{-}(\Gamma\text{−Free}) \simeq K^{-}(\text{Sum}-T) \longrightarrow K^{-}(\Lambda\text{−Proj})$$

is fully faithful.

Our proof of Theorem 4.5.1 diverges from Rickard's original proof [Ric89, Theorem 3.5] and is centered around the following Lemma 4.5.2. This Lemma is a variant of Keller's principle of infinite dévissage [KZ98, 8.1.3]; basically it tells us that by "infinite unscrewing" any complex can be reduced to complexes with just the regular module sitting in a single degree.

Lemma 4.5.2 (Dévissage)**.** *Let* Γ *be any ring. If a full triangulated subcategory* $\mathcal{V}$ *of* $D^{-}(\Gamma) \simeq K^{-}(\Gamma\text{−Free})$ *is closed under all the direct sums that exist in* $K^{-}(\Gamma\text{−Free})$ *and contains* $\lceil\Gamma\rfloor$*, then* $\mathcal{V}$ *is already equal to the whole category* $K^{-}(\Gamma\text{−Free})$.

Proof. First of all, we prove by induction on $n \geq 0$ that any bounded complex $X = X_{-n} \to \cdots \to X_0$ lies in $\mathcal{V}$. The induction start for $n = 0$ is true by assumption, since X_0 being a free module is a sum of copies of Γ.
For the induction step $n-1 \to n$ consider the standard truncation triangle (see Example B.3.4)

$$X\big|_{n-1} \to X \to \lceil X_{-n}\rfloor_{-n} \rightsquigarrow,$$

where $X\big|_{n-1}$ is the truncated complex $X_{-n-1} \to \cdots \to X_0$ and lies in $\mathcal{V}$ by induction. Since X_{-n} is a free module, the rightmost term is a direct sum of copies of $\lceil\Gamma\rfloor_{-n}$, hence it also lies in $\mathcal{V}$. This finishes the induction, since $\mathcal{V}$ being a triangulated subcategory is closed under extending triangles and therefore contains X as well.

Next we write an arbitrary (right-bounded) complex X as the direct limit of the system

$$X\big|_0 \to X\big|_1 \to X\big|_2 \to \cdots$$

of truncated subcomplexes of X. This gives an exact sequence of (right-bounded) complexes

$$0 \to \bigoplus_{i\in\mathbb{N}} X\big|_i \to \bigoplus_{j\in\mathbb{N}} X\big|_j \to X \to 0,$$

where the first arrow is induced by the maps $X\big|_i \xrightarrow{\binom{+}{-}} X\big|_i \oplus X\big|_{i+1}$.

Finally we pass to the derived category and obtain a triangle

$$\bigoplus_{i\in\mathbb{N}} X\big|_i \to \bigoplus_{j\in\mathbb{N}} X\big|_j \to X \rightsquigarrow,$$

where the first two terms are in $\mathcal{V}$ since they are direct sums of bounded complexes. Thus X is in $\mathcal{V}$ as well, which concludes the proof. □

Lemma 4.5.3. *Let $F\colon \mathcal{T} \to \mathcal{S}$ be a triangulated functor of triangulated categories and let Y be some object of $\mathcal{T}$. Let ff_Y^F denote the full subcategory of $\mathcal{T}$ where F is fully faithful on maps into Y, i.e. it consists of those objects X such that for all integers n the map induced by F*

$$\mathrm{Hom}_{\mathcal{T}}(X[n], Y) \longrightarrow \mathrm{Hom}_{\mathcal{S}}(FX[n], FY)$$

is an isomorphism. Dually, let ff_F^Y the full subcategory of $\mathcal{T}$, where F is fully faithful on maps from Y.
Then both ff_Y^F and ff_F^Y are triangulated subcategories of $\mathcal{T}$.
Moreover, if F preserves direct sums, then ff_Y^F is closed under direct sums and direct summands. Dually, if F preserves products, then ff_F^Y is closed under factors and products.

Proof. Since the statements about ff_Y^F and ff_F^Y are dual, we restrict our attention to the former. It is already built into the definition that ff_Y^F is closed under the suspension functor so, to see that it is a triangulated subcategory, we only need to check that it is closed under completing triangles. If we have a triangle $X' \to X \to X'' \rightsquigarrow$ we get, by applying F, another triangle $FX' \to FX \to FX'' \rightsquigarrow$. Therefore by applying $\mathrm{Hom}_{\mathcal{T}}(-, Y)$ respectively $\mathrm{Hom}_{\mathcal{S}}(-, FY)$ we get for each integer n a diagram with exact columns as follows:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{T}}(X''[n-1], Y) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{S}}(FX''[n-1], FY) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{T}}(X'[n], Y) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{S}}(FX'[n], FY) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{T}}(X[n], Y) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{S}}(FX[n], FY) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{T}}(X''[n], Y) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{S}}(FX''[n], FY) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathcal{T}}(X'[n+1], Y) & \xrightarrow{F} & \mathrm{Hom}_{\mathcal{S}}(FX'[n+1], FY)
\end{array}$$

This diagram commutes since F is a functor and therefore by the 5-Lemma we see that X is in ff_Y^F if X' and X'' are.
Hence ff_Y^F is a triangulated subcategory of $\mathcal{T}$.

We are left to prove the statement about direct sums and direct summands.
For all $n \in \mathbb{Z}$ and any family $(X_i)_{i\in I}$ of objects in $\mathcal{T}$ that has a direct sum, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}\left(\bigoplus_i X_i[n], Y\right) & \xrightarrow{\cong} & \prod_i \mathrm{Hom}_{\mathcal{T}}\left(X_i[n], Y\right) \\ \big\downarrow F & & \big\downarrow \prod_i F \\ \mathrm{Hom}_{\mathcal{S}}\left(F\bigoplus_i X_i[n], FY\right) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathcal{S}}\left(FX_i[n], FY\right) \end{array}.$$

If F commutes with direct sums, the lower horizontal arrow is an isomorphism; therefore we see that $\bigoplus_i X_i$ is in ff_Y^F if and only if all the X_i's are, which means precisely that ff_Y^F is closed under direct sums and direct summands. □

Proof of Theorem 4.5.1. The strategy of the proof will be as follows:
First we show that for any object Y in $K^-(\Gamma-\mathrm{Free})$ and any integer n, the map

$$\mathrm{Hom}_{K^-(\Gamma-\mathrm{Free})}\left(\lceil\Gamma\rfloor[n], Y\right) \longrightarrow \mathrm{Hom}_{K^-(\Lambda-\mathrm{Proj})}\left(F\lceil\Gamma\rfloor[n], FY\right)$$

induced by F is an isomorphism. Furthermore, we shall prove that F commutes with arbitrary direct sums. With the notation of Lemma 4.5.3 (and using that same lemma), we can then conclude that the triangulated subcategory $\mathrm{ff}_Y^F \subseteq K^-(\Gamma-\mathrm{Free})$ contains $\lceil\Gamma\rfloor$ and is closed under direct sums. Therefore, for all objects Y of $K^-(\Gamma-\mathrm{Free})$ we have an equality $\mathrm{ff}_Y^F = K^-(\Gamma-\mathrm{Free})$ by Lemma 4.5.2; this means precisely that F is fully faithful.

To start off the proof of the first step, notice that by replacing Y with $Y[-n]$ we may without loss of generality assume $n = 0$.

First of all, we consider the case where Y is just concentrated in one degree, i.e. of the form $\lceil\Gamma^{(I)}\rfloor_m$ for some index set I and integer m.
We identify Γ with T and therefore Y with $\lceil T^{(I)}\rfloor_m$ via the equivalence $\Gamma-\mathrm{Free} \simeq \mathrm{Sum}-T$ of Lemma 4.1.1. Hence we have to show that

$$F\colon\ \mathrm{Hom}\left(\lceil T\rfloor, \lceil T^{(I)}\rfloor_m\right) \longrightarrow \mathrm{Hom}\left(F\lceil T\rfloor, F\lceil T^{(I)}\rfloor_m\right)$$

is an isomorphism.
For $m = 0$ this is true, since by Remark 4.4.1 we know that the restriction of F to

Sum$-T \xhookrightarrow{\lceil - \rfloor_0} K^-$ (Sum$-T$) is just the inclusion of the full subcategory Sum$-T \subset K^-(\Lambda-$Proj$)$, hence fully faithful.
If $m \neq 0$, then Remark 4.4.1 at least tells us that $F\lceil T^{(I)} \rfloor_m = T^{(I)}[-m]$ and $F\lceil T \rfloor = T$. Since T is a tilting complex there can be no nontrivial homomorphism between them. On the other hand, Hom $\left(\lceil T \rfloor, \lceil T^{(I)} \rfloor_m\right)$ is clearly zero as well, since we are looking at two complexes concentrated in degrees 0 and $m \neq 0$ respectively. Hence in this case F is an isomorphism for trivial reasons.

If $Y = Y_{m-k} \to \cdots \to Y_m$ is a bounded chain complex, we can prove the statement by induction on $k \in \mathbb{N}$ using the standard triangles $Y\big|_{k-m-1} \to Y \to \lceil Y_{k-m} \rfloor_{k-m} \rightsquigarrow$ and the long exact Hom-sequence.

Now we need to see what happens if Y is unbounded. Let N be a large natural number and consider the canonical inclusion $Y' := Y\big|_N \overset{\iota}{\hookrightarrow} Y$ of the complex truncated at the index $-N$. This induces a commutative diagram as follows

$$\begin{array}{ccc} \mathrm{Hom}_{K^-}\left(\lceil \Gamma \rfloor, Y'\right) & \xrightarrow{F} & \mathrm{Hom}_{K^-}\left(F\lceil \Gamma \rfloor, FY'\right) \\ \downarrow{\scriptstyle \iota_\star} & & \downarrow{\scriptstyle (F\iota)_\star} \\ \mathrm{Hom}_{K^-}\left(\lceil \Gamma \rfloor, Y\right) & \xrightarrow{F} & \mathrm{Hom}_{K^-}\left(F\lceil \Gamma \rfloor, FY\right) \end{array},$$

where the top horizontal arrow is an isomorphism by what we just proved about bounded complexes such as Y'. It is therefore enough to show that the vertical arrows are isomorphisms as well.
Since $\lceil \Gamma \rfloor$ is concentrated in degree zero, the chain homomorphisms $\lceil \Gamma \rfloor \to Y$ depend only on the non-negative degrees of Y. Since we must also check for homotopies we must also consider Y_{-1}, but for N large enough Y' and Y agree in all relevant components; therefore the vertical map on the left is an isomorphism.
Similarly, notice that $F\lceil \Gamma \rfloor$ (which is nothing but T) is bounded by some indices, without loss of generality we can assume them to be 1 and r. Therefore, to get that the vertical map on the right is an isomorphism we need to show that we can make FY' and FY agree for all nonnegative degrees by choosing a sufficiently large cut-off point N.
After identifying $Y \in K^-(\Gamma-$Free$)$ with the corresponding object in K^- (Sum$-T$), the k-th component of FY is $\bigoplus_{i \in \mathbb{Z}} Y^{i,k-i}$. Since $Y^{i,k-i}$ is nonzero only for $1 \leq i \leq r$, this sum depends only on the components of Y in the range from $k-r$ to $k-1$. Hence the nonnegative components of FY depend only on $Y\big|_r$ and we can choose $N = r$. Now we just need to see that the differentials of FY and FY' agree in that range too, but this is easy since we can obtain a valid augmentation of the differentials of Y' by restricting the augmentation of Y.

The last thing we need to do is understand what F does with direct sums. For a family $(X_i)_{i \in I}$ of objects in $K^-(\mathrm{Sum}{-}T)$, the Λ-grid $X^{*,*}$ associated to $X = \bigoplus_i X_i$ is certainly just the direct sum of all the $X_i^{*,*}$. Moreover, the augmented differential in $X^{*,*}$ can be chosen as the direct sum of all the individual augmented differentials. Finally, we observe that the functor tot respects all these direct sums, as its components are some finite direct sums of the original Λ-grid and all direct sums commute with each other.
Therefore F commutes with arbitrary direct sums and the proof of Theorem 4.5.1 is completed. $\square$

5. T-resolutions and the functor G

We keep the set up of the last section: Λ is some ring, T is a tilting complex over Λ and $\Gamma := \operatorname{End}^{\mathrm{opp}}_{K^-(\Lambda)}(T)$ is its opposite endomorphism ring.
In this section we want to define an inverse $G\colon K^-(\Lambda\text{−Proj}) \to K^-(\text{Sum}-T)$ to the functor F. The idea is to define for any object X of $K^-(\Lambda\text{−Proj})$ a "T-resolution"

$$\cdots \to U^{(N-2)} \to U^{(N-1)} \to U^{(N)} \to X[N] \to 0, \tag{5.1}$$

where the $U^{(n)}$ are sums of copies of T and N is some suitable integer. In analogy to free resolutions of modules we will then replace X by the complex formed by the $U^{(n)}$'s. Since we do not have the notion of an of exact sequence in the triangulated (and not abelian) category $K^-(\Lambda\text{−Proj})$, we will have to find some alternative condition to put on the sequence (5.1).

5.1. Construction of T-resolutions

Let X be an object in $K^-(\Lambda\text{−Proj})$. Since T is bounded, $\operatorname{Hom}_{K^-}(T, X[n])$ is zero for all sufficiently large n, because we can shift X so far to the left that the supports of T and $X[n]$ do not overlap anymore. We can then define N to be the largest integer n such that $\operatorname{Hom}_{K^-}(T, X[n])$ doesn't vanish. If $\operatorname{Hom}_{K^-}(T, X[n])$ is always zero, just set $N := 0$.
Now, we recursively define sequences $\left(X^{(i)}\right)_{i\le N}$ and $\left(U^{(i)}\right)_{i\le N}$ of objects in $K^-(\Lambda\text{−Proj})$ together with triangles

$$X^{(i-1)} \to U^{(i)} \to X^{(i)} \rightsquigarrow \tag{5.2}$$

as follows:

1. $X^{(N)} := X[N]$.
2. If $X^{(i)}$ is already defined, then we define $U^{(i)} \to X^{(i)}$ by setting

   $$U^{(i)} := \bigoplus_{T\to X^{(i)}} T,$$

   where the direct sum is indexed by all maps $f \in \operatorname{Hom}_{K^-}\left(T, X^{(i)}\right)$ and the f-th component of the map $U^{(i)} \to X^{(i)}$ is just f.
3. If $U^{(i)}$ and $X^{(i)}$ are already defined as in step 2, then we complete $U^{(i)} \to X^{(i)}$ to the triangle (5.2) to get $X^{(i-1)}$. Of course $X^{(i-1)} \to U^{(i)}$ is only uniquely determined up to isomorphism of triangles but we just choose any such triangle (for instance a shifted mapping cone) and stick with it for the rest of the construction.

Finally, we put all the $U^{(i)}$'s together to form a complex $U = U_X$:

$$\cdots \xrightarrow{d_{N-3}} U^{(N-2)} \xrightarrow{d_{N-2}} U^{(N-1)} \xrightarrow{d_{N-1}} U^{(N)} \tag{5.3}$$

by choosing the differential d_{i-1} to be the composition

$$U^{(i-1)} \to X^{(i-1)} \to U^{(i)},$$

where both maps come from the triangles (5.2). This is a chain complex, since when composing two differentials, two consecutive maps in the triangle (5.2) appear.
The complex U will be called a **T-resolution** of X; the goal of this section will be to show that this construction gives a well-defined functor

$$G\colon K^-(\Lambda\text{–Proj}) \longrightarrow K^-(\text{Sum–}T),$$

which maps X to U_X and is a quasi-inverse of the functor F.

Before we go on, let us prove a useful property of T-resolutions.

Lemma 5.1.1. *The group* $\operatorname{Hom}\left(T, X^{(i)}[n]\right)$ *is zero for all* $i \leq N$ *and all positive integers* n.

Proof. We do a downward induction starting from $i = N$.
The claim is obvious for $i = N$, since $X^{(N)}[n] = X[N][n] = X[N+n]$ and N was chosen maximal with $\operatorname{Hom}(T, X[N]) \neq 0$.
To prove the induction step $i \to i-1$ we apply $\operatorname{Hom}(T, -)$ to the triangle (5.2) and get a long exact sequence, of which we consider the following two segments:

$$\operatorname{Hom}\left(T, U^{(i)}\right) \twoheadrightarrow \operatorname{Hom}\left(T, X^{(i)}\right) \to \operatorname{Hom}\left(T, X^{(i-1)}[1]\right) \to \operatorname{Hom}\left(T, U^{(i)}[1]\right) \to \tag{5.4}$$

$$\cdots \to \operatorname{Hom}\left(T, X^{(i)}[n-1]\right) \to \operatorname{Hom}\left(T, X^{(i-1)}[n]\right) \to \operatorname{Hom}\left(T, U^{(i)}[n]\right) \to \cdots. \tag{5.5}$$

The first arrow is clearly a surjection by the definition of $U^{(i)}$, since every $g \in \operatorname{Hom}\left(T, X^{(i)}\right)$ is hit by the map $\iota_g\colon T \to \bigoplus_f T = U^{(i)}$. Moreover, $U^{(i)}$ is a sum of copies of T, which is a tilting complex; hence $\operatorname{Hom}\left(T, U^{(i)}[1]\right)$ vanishes by T1. Therefore the sequence (5.4) gives $\operatorname{Hom}\left(T, X^{(i-1)}[1]\right) = 0$.
To see that $\operatorname{Hom}\left(T, X^{(i-1)}[n]\right) = 0$ for $n \geq 2$ we use the sequence (5.5). Again, $\operatorname{Hom}\left(T, U^{(i)}[n]\right)$ is zero by T1 and $\operatorname{Hom}\left(T, X^{(i)}[n-1]\right)$ is zero by induction. □

5.2. F has a right adjoint G given by T-resolutions

From now on we will simplify the notation by changing the indexing of the complex X such that $N = 0$. Once again, fix a T-resolution U of X.

Lemma 5.2.1. *For any positive integer n we have a triangle*

$$X^{(-n)}[n-1] \xrightarrow{g_n} F\left(U\big|_{n-1}\right) \xrightarrow{\epsilon^n} X \rightsquigarrow \tag{5.6}$$

such that the ϵ^n are compatible with truncation, i.e. the following diagram commutes for all $n \geq 2$:

$$\begin{array}{ccc} F\left(U\big|_{n-1}\right) & \searrow^{\epsilon^n} & \\ \uparrow & & X \quad . \\ F\left(U\big|_{n-2}\right) & \nearrow_{\epsilon^{n-1}} & \end{array} \tag{5.7}$$

Proof. We proceed by induction on n and carry along the commutativity of the following diagram:

$$\begin{array}{ccc} U^{(-n)}[n-1] & \longrightarrow & X^{(-n)}[n-1] \\ \| & & \downarrow g_n \\ F\left(\lceil U^{(-n)} \rfloor_{1-n}\right) & \xrightarrow[F(d_{-n})]{} & F\left(U\big|_{n-1}\right) \end{array} \quad . \tag{5.8}$$

The identification $U^{(-n)}[n-1] = F\left(\lceil U^{(-n)} \rfloor_{1-n}\right)$ is described in Remark 4.4.1. Note that we view the differential $d_{-n} \colon U^{(-n)} \to U^{(-n+1)}$ as a map of complexes $\lceil U^{(-n)} \rfloor_{-n+1} \to U\big|_{n-1}$ so that we can apply F to it.

Let's start with the case $n = 1$. Observe that X is simply $X^{(0)}$ since we are assuming $N = 0$; by Remark 4.4.1 we see that $F\left(U\big|_0\right) = F\left(\lceil U^{(0)} \rfloor\right)$ is nothing but $U^{(0)}$. Therefore the required triangle (5.6) can just be taken to be the defining triangle (5.2) of $X^{(-1)}$. The square in (5.8) just asserts that the differential

$$d_{-1} \colon F\left(\lceil U^{(-1)} \rfloor_0\right) = U^{(-1)} \longrightarrow U^{(0)} = F\left(U\big|_0\right)$$

of U is given by $U^{(-1)} \to X^{(-1)} \to U^{(0)}$, which is true by definition. For the diagram (5.7) there is nothing to show yet.

To do the induction step $n \to n+1$ we look at the following diagram:

$$\begin{array}{ccccccc}
U^{(-n)}[n-1] & \longrightarrow & X^{(-n)}[n-1] & \longrightarrow & X^{(-(n+1))}[n] & \longrightarrow & U^{(-n)}[n] \\
\| & & \downarrow{\scriptstyle g_n} & & \vdots{\scriptstyle g_{n+1}} & & \| \\
F\left(\lceil U^{(-n)} \rfloor_{-n}[-1]\right) & \xrightarrow{F(d_{-n})} & F\left(U\big|_{n-1}\right) & \longrightarrow & F\left(U\big|_{n}\right) & \longrightarrow & F\left(\lceil U^{(-n)} \rfloor_{-n}\right) \\
 & & \downarrow{\scriptstyle \epsilon^n} & & \vdots{\scriptstyle \epsilon^{n+1}} & & \\
0 & \longrightarrow & X & \xrightarrow{\mathbb{1}} & X & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \\
U^{(-n)}[n] & \longrightarrow & X^{(-n)}[n] & \longrightarrow & X^{(-(n+1))}[n+1] & \longrightarrow & U^{(-n)}[n+1]
\end{array}$$

The first row is the triangle (5.2) and the second row arises by applying the triangulated functor F to the standard triangle

$$U\big|_{n-1} \to U\big|_{n} \to \lceil U^{(-n)} \rfloor_{-n} \rightsquigarrow .$$

Furthermore, the second column is the triangle (5.6) and the leftmost square is the commutative diagram (5.8), both given by induction. The octahedral axiom [Nee01, Proposition 1.4.6] tells us that we can fill in the dashed arrows in such a way, that everything commutes and that the third column is a triangle. This triangle is exactly of the form (5.6) we were looking for and the central square implies the commutativity of the diagram (5.7). Therefore, the only thing left to check is the commutativity of the square (5.8) (with n replaced by $n+1$).

The right hand square in the diagram

$$\begin{array}{ccccc}
U^{(-n-1)}[n] & \longrightarrow & X^{(-n-1)}[n] & \longrightarrow & U^{(-n)}[n] \\
\| & & \downarrow{\scriptstyle g_{n+1}} & & \| \\
F\left(\lceil U^{(-n-1)} \rfloor_{-n}\right) & \xrightarrow{F(d_{-n-1})} & F\left(U\big|_{n}\right) & \xrightarrow{F(\mathrm{can})} & F\left(\lceil U^{(-n)} \rfloor_{-n}\right)
\end{array}$$

is commutative, since it appears in the commutative diagram we got from the octahedral axiom. Furthermore, the outer rectangle commutes since both horizontal compositions are given by the differential $d_{-n-1} \colon U^{(-n-1)} \to U^{(-n)}$. Note that the shift by $[n]$ in the top row corresponds exactly to the fact that we have complexes concentrated in degree $-n$ in the bottom row. Hence we get that the square (5.8) becomes commutative after postcomposing with $F\,(\mathrm{can})$, which is almost, but not quite, what we wanted.

To prove that we can omit this postcomposition and still get a commutative square, we look at the following commutative diagram of Hom-sets

$$\begin{array}{ccc} \mathrm{Hom}\left(\left\lceil U^{(-n-1)}\right\rfloor_{-n}, U\big|_{n}\right) & \xrightarrow{\quad \mathrm{can}_\star \quad} & \mathrm{Hom}\left(\left\lceil U^{(-n-1)}\right\rfloor_{-n}, \left\lceil U^{(-n)}\right\rfloor_{-n}\right) \\ \cong\big\downarrow F & & \cong\big\downarrow F \\ \mathrm{Hom}\left(F\left(\left\lceil U^{(-n-1)}\right\rfloor_{-n}\right), F\left(U\big|_{n}\right)\right) & \xrightarrow{F(\mathrm{can})_\star} & \mathrm{Hom}\left(F\left(\left\lceil U^{(-n-1)}\right\rfloor_{-n}\right), F\left(\left\lceil U^{(-n)}\right\rfloor_{-n}\right)\right) \end{array},$$

where the vertical maps are isomorphism since F is fully faithful by Theorem 4.5.1. It is easy to see that the top map is injective, since every morphism $\left\lceil U^{(-n-1)}\right\rfloor_{-n} \to U\big|_{n}$ is determined by the restriction to the support of its domain, i.e. by its component $U^{(-n-1)} \to U^{(-n)}$. We conclude that the lower arrow $F(\mathrm{can})_\star$ is injective as well, which implies that we can cancel the unwanted postcomposition with $F(\mathrm{can})$ and get the desired result. □

Proposition 5.2.2. *Let X be an object in $K^-(\Lambda{-}\mathrm{Proj})$ and let U_X in $K^-(\mathrm{Sum}{-}T)$ be a projective resolution of X. Then there exists a map $\epsilon_X\colon FU_X \to X$ such that the induced map*

$$\epsilon_X \circ -\colon \mathrm{Hom}_{K^-(\Lambda-\mathrm{Proj})}(FQ, FU_X) \longrightarrow \mathrm{Hom}_{K^-(\Lambda-\mathrm{Proj})}(FQ, X)$$

is an isomorphism for all objects Q in $K^-(\mathrm{Sum}{-}T)$.

For simplicity of notation, just write U instead of U_X. As the notation suggests, we would like to glue together the maps $\epsilon^n\colon F\left(U\big|_{n-1}\right) \to X$ appearing in the triangles (5.6) to obtain the desired $\epsilon\colon FU \to X$. The compatibility condition (5.7) is clearly necessary for this glueing process and in fact it will turn out to be sufficient as well. However, since we are working with chain complexes modulo homotopy and not with actual chain complexes, things are not quite as easy as we might hope for. Therefore, before going on we make a brief digression and prove the following

Lemma 5.2.3. *Fix an additive category $\mathcal{A}$ and let $Y^0 \xrightarrow{\sigma^0} Y^1 \xrightarrow{\sigma^1} \cdots$ be a direct system in $C^b(\mathcal{A})$. Assume that for any individual homological degree $l \in \mathbb{Z}$, the corresponding direct system $Y^0_l \xrightarrow{\sigma^0_l} Y^1_l \xrightarrow{\sigma^1_l} \cdots$ of objects in $\mathcal{A}$ becomes stationary, in the sense that $Y^n_l \xrightarrow{\sigma^n_l} Y^{n+1}_l$ is an isomorphism for sufficiently large n.*
Let $Y := \lim\limits_{n\to} Y^n$ be the the direct limit in $C(\mathcal{A})$ and let X be any (not necessarily bounded) complex with entries in $\mathcal{A}$. Then the canonical map

$$\mathrm{Hom}_K(Y, X) \longrightarrow \lim_{\leftarrow n} \mathrm{Hom}_K(Y^n, X)$$

of Hom-groups of the homotopy category is surjective.

Remark 5.2.1. (i) The direct system $Y^\bullet$ has entries in the additive category of bounded chain complexes. This means that each Y^n (for $n \in \mathbb{N}$) is a bounded complex with entries $Y_l^n \in \operatorname{Ob}\mathcal{A}$ (for $l \in \mathbb{Z}$) and that the $\sigma^n \colon Y^n \to Y^{n+1}$ are chain maps with components $\sigma_l^n \colon Y_l^n \to Y_l^{n+1}$. We can turn this description around and view the object $Y^\bullet$ as a chain complex $\cdots \to Y_l^\bullet \to Y_{l+1}^\bullet \cdots$ of direct systems $Y_l^0 \xrightarrow{\sigma_l^0} Y_l^1 \xrightarrow{\sigma_l^1} \cdots$ in $\mathcal{A}$.

(ii) The surjectivity of the map $\operatorname{Hom}_K(Y, X) \to \varprojlim_n \operatorname{Hom}_K(Y^n, X)$ expresses precisely that we can glue maps that are compatible up to homotopy. Indeed, an element of $\varprojlim_n \operatorname{Hom}_K(Y^n, X)$ is just a collection $(f^n \colon Y^n \to X)_{n\in\mathbb{N}}$ of chain maps, such that the diagram

$$\begin{array}{ccc} Y^n & \xrightarrow{f^n} & \\ \downarrow & & X \\ Y^{n+1} & \xrightarrow{f^{n+1}} & \end{array} \tag{5.9}$$

commutes in the homotopy category for all $n \in \mathbb{N}$. A preimage of such a collection under the canonical map is nothing but a chain map $f \colon Y \to X$, such that for all $n \in \mathbb{N}$ the composition $Y^n \to Y \xrightarrow{f} X$ is homotopic to f^n.

(iii) The statement of Lemma 5.2.3 is not at all a tautology. The fact that Y is the limit of the system $(Y^0 \to Y^1 \to \cdots)$ in the category of *strict* chain complexes means by definition that the canonical map

$$\operatorname{Hom}_C(Y, X) \longrightarrow \varprojlim_n \operatorname{Hom}_C(Y^n, X)$$

(note the subscript C of the Hom-sets) is an isomorphism. This means that we can (uniquely) glue systems $(f^n \colon Y^n \to X)_{n\in\mathbb{N}}$ of chain maps when the diagram (5.9) commutes on the nose. It is not a priori clear (and in fact false in general) that we can glue maps that are only compatible *up to homotopy.*

(iv) The complex (Y, d) can be described explicitly in terms of the complexes (Y^j, d^j) as having entries $Y_l := Y_l^m$ and differentials $d_l := d_l^m$, where m is sufficiently large and depends on l. This is well-defined up to the isomorphisms $Y_l^n \xrightarrow{\sigma_l^n} Y_l^{n+1}$ in the stable range of the system $(Y_l^0 \to Y_l^1 \to \cdots)$.

For the proof of this lemma we use the Mittag-Leffler condition on inverse systems of abelian groups. See Appendix C for more details.

Proof of Lemma 5.2.3. The main trick is to define for any pair of chain complexes V and W a new chain complex $\mathbf{Hom}(V, W)$ with

- entries $\mathbf{Hom}_i\left(V,W\right)=\prod_{l\in\mathbb{Z}}\operatorname{Hom}\left(V_l,W_{l+i}\right)$ and
- differentials $d_i\colon\ (f_l)_{l\in\mathbb{Z}}\mapsto\left(f_{l+1}d_l^V+(-1)^i\,d_{l+i}^W f_l\right)_{l\in\mathbb{Z}}$.

This is a well-defined complex (the differential squares to zero) because of the choice of sign in the differential. Moreover an easy calculation shows that the 0-th homology $H_0\,\mathbf{Hom}\left(V,W\right)$ of this complex is nothing but the Hom-set $\operatorname{Hom}_K\left(V,W\right)$ in the homotopy category.
Now we consider the inverse system $A^\bullet$ of complexes of abelian groups given as

$$\mathbf{Hom}\left(Y^0,X\right)\leftarrow\mathbf{Hom}\left(Y^1,X\right)\leftarrow\cdots,$$

which limits to the complex $\lim\limits_{\leftarrow n}A^n=\mathbf{Hom}\left(Y,X\right)$ since Y is the direct limit of the system $Y^\bullet$. Proposition C.4 gives us the desired surjectivity of the map

$$\operatorname{Hom}_K\left(Y,X\right)=H_0\lim_{\leftarrow n}A^n\longrightarrow\lim_{\leftarrow n}H_0A^n=\lim_{\leftarrow n}\operatorname{Hom}_K\left(Y^n,X\right)$$

if we can prove that the component inverse system $A^\bullet_{-1}$ is Mittag-Leffler, i.e. that the image of the map $A^n_{-1}\leftarrow A^{n+m}_{-1}$ stops changing eventually for $m\to\infty$.
Fix a natural number n and observe that for any m we can factor the map $A^n_{-1}\leftarrow A^{n+m}_{-1}$ as

$$\begin{array}{ccc}A^n_{-1}=\prod\limits_{l\in\mathbb{Z}}\operatorname{Hom}\left(Y^n_l,X_{l-1}\right) & \longleftarrow & \prod\limits_{l\in\mathbb{Z}}\operatorname{Hom}\left(Y^{n+m}_l,X_{l-1}\right)=A^{n+m}_{-1}\\ & \nwarrow & \big\downarrow \mathrm{pr}\\ & & \prod\limits_{l\in\operatorname{supp}Y^n}\operatorname{Hom}\left(Y^{n+m}_l,X_{l-1}\right)\end{array}.$$

Since the vertical projection is surjective, the image of $A^n_{-1}\leftarrow A^{n+m}_{-1}$ depends only on (the image of) the dashed arrow. Every single factor of the product in the lower right stops changing eventually for $m\to\infty$, because each individual component system $Y^\bullet_l$ becomes stationary by assumption. Moreover, there are only finitely many such factors (since Y^n is a bounded complex), which means that the domain of the dashed arrow and hence the arrow itself stop changing eventually. We have therefore proved that $A^\bullet_{-1}$ is Mittag-Leffler, hence the proof is concluded. □

Proof of Proposition 5.2.2. The first thing we have to do is to check that the assumptions of Lemma 5.2.3 are satisfied for the directed system

$$F\left(U\big|_0\right)\to F\left(U\big|_1\right)\to F\left(U\big|_2\right)\to\cdots \tag{5.10}$$

and that FU is really the direct limit of this system. As announced previously and

using Remark 5.2.1 (ii), we can then glue the morphisms $\epsilon^n \colon F\left(U\big|_{n-1}\right) \to X$ appearing in the triangles (5.6) to obtain a map $\epsilon \colon FU \to X$.
Let $U^{*,*}$ be the Λ-grid underlying U and let d be a choice of augmented differential extending the differentials of U. We can truncate this augmented bicomplex vertically below $-n$ by replacing $U^{i,j}$ by zero if $j < -n$ and restricting d accordingly. This procedure gives again an augmented bicomplex, which is then a possible choice for the image of $U\big|_n$ in $\mathfrak{B}(\Lambda)$. For any index l, the l-th components of the maps $F\left(U\big|_n\right) \to F\left(U\big|_{n+1}\right) \to FU$ are just

$$\bigoplus_{\substack{i+j=l\\ j\geq -n}} U^{i,j} \longrightarrow \bigoplus_{\substack{i+j=l\\ j\geq -n-1}} U^{i,j} \longrightarrow \bigoplus_{i+j=l} U^{i,j}.$$

Since those sums are actually finite (due to the fact that T is bounded and so $U^{*,*}$ is horizontally bounded), the maps are the identity for n large enough. Clearly, this identities are compatible with the differentials of FU, $F\left(U\big|_n\right)$ and $F\left(U\big|_{n+1}\right)$ because the augmented differentials of U, $U\big|_n$ and $U\big|_{n+1}$ agree on every nonzero component of the Λ-grid $U^{*,*}$. We have therefore proved that the directed system (5.10) stabilizes componentwise and converges to FU, which is what we need to apply Lemma 5.2.3.

Next, we set up a dévissage argument to prove that ϵ induces the desired isomorphism on Hom-sets. Let $\mathcal{V}$ be the full subcategory of $K^-(\mathrm{Sum}{-}T)$ consisting of those objects Q, such that the map

$$\epsilon \circ - \colon \operatorname{Hom}_{K^-(\Lambda-\mathrm{Proj})}(FQ[k], FU) \longrightarrow \operatorname{Hom}_{K^-(\Lambda-\mathrm{Proj})}(FQ[k], X) \tag{5.11}$$

is an isomorphism for all integers k.
Using the fact that F is triangulated and commutes with arbitrary direct sums (see the end of the proof of Theorem 4.5.1), it is easy to see that $\mathcal{V}$ is a triangulated subcategory of $K^-(\mathrm{Sum}{-}T)$ and closed under direct sums (the argument is very similar to the proof of Lemma 4.5.3 and is therefore omitted). Hence, all we need to do is check that $\lceil T \rfloor$ lies in $\mathcal{V}$, since then we can use dévissage to deduce $\mathcal{V} = K^-(\mathrm{Sum}{-}T)$, which is what we want to prove. To apply the dévissage principle as stated in Lemma 4.5.2, recall that we identify $K^-(\mathrm{Sum}{-}T)$ with $K^-(\Gamma{-}\mathrm{Free})$ via the functor $K^-(\mathbb{T})$, which maps $\lceil T \rfloor$ to $\lceil \Gamma \rfloor$.
To see that $\lceil T \rfloor$ is in $\mathcal{V}$, fix an integer k. Observe that $F\lceil T \rfloor[k]$ is nothing but $T[k]$

by Remark 4.4.1, hence the arrow (5.11) appears in the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{K^-}\left(T[k], FU\right) & \xleftarrow{\left(Fi^{n-1}\right)\circ -} & \mathrm{Hom}_{K^-}\left(T[k], F\left(U\big|_{n-1}\right)\right) \\ & \searrow^{\epsilon \circ -} & \downarrow{\scriptstyle \epsilon^n \circ -} \\ & & \mathrm{Hom}_{K^-}\left(T[k], X\right) \end{array}, \tag{5.12}$$

where $i^{n-1}\colon U\big|_{n-1} \to U$ is the canonical inclusion. The diagram (5.12) commutes, since $\epsilon^n = \left(Fi^{n-1}\right)\epsilon$ by the choice of the map ϵ. We need to show that the diagonal map is an isomorphism, hence it suffices to prove that the other two maps are isomorphisms for n large enough.

Since $T[k]$ is bounded, maps $T[k] \to FU$ only depend on a bounded region of the complex FU. Moreover, the system $F\left(U\big|_0\right) \to F\left(U\big|_1\right) \to \cdots$ (which converges to FU) stabilizes in each individual degree, hence on this bounded region the map $F\left(U\big|_{n-1}\right) \to FU$ is eventually an isomorphism. This proves that $\left(Fi^{n-1}\right)\circ -$ is eventually bijective.

To prove that $\epsilon^n \circ -$ is bijective as well, we apply $\mathrm{Hom}_{K^-}\left(T[k], -\right)$ to the triangle (5.6) and obtain an exact sequence

$$\begin{aligned} &\mathrm{Hom}\left(T, X^{(-n)}[n-1-k]\right) \to \mathrm{Hom}\left(T[k], F\left(U\big|_{n-1}\right)\right) \xrightarrow{\epsilon^n \circ -} \mathrm{Hom}\left(T[k], X\right) \\ &\qquad \to \mathrm{Hom}\left(T, X^{(-n)}[n-k]\right). \end{aligned}$$

If n is chosen so large that $n-1-k > 0$ (hence $n-k>0$) and $-n < N$ (where N is the constant coming from the construction of a T-resolution), then Lemma 5.1.1 tells us that the two outer terms in the exact sequence vanish and so $\epsilon^n \circ -$ is an isomorphism. □

Corollary 5.2.4. *The functor* $F\colon K^-(\mathrm{Sum}{-}T) \to K^-(\Lambda{-}\mathrm{Proj})$ *has a right adjoint*

$$G\colon K^-(\Lambda{-}\mathrm{Proj}) \longrightarrow K^-(\mathrm{Sum}{-}T)$$

which is given on objects by $X \mapsto U_X$*.*

Proof. Since F induces bijections on Hom-sets (i.e. is fully faithful) by Theorem 4.5.1, we obtain the desired result as an immediate consequence of Proposition 5.2.2 and Lemma 5.2.5 below. □

The following lemma is a well known characterization of adjunctions, hence we omit its proof.

Lemma 5.2.5. *[ML98, §IV.1 Theorem 2 (iv)] Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a functor between* *two arbitrary categories such that for any X in $\mathcal{D}$ we find an universal arrow from F to X, i.e. an object U_X of $\mathcal{C}$ together with an arrow $\epsilon_X \colon FU_X \to X$ such that the composition*

$$\mathrm{Hom}_{\mathcal{C}}(Q, U_X) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(FQ, FU_X) \xrightarrow{\epsilon_X \circ -} \mathrm{Hom}_{\mathcal{D}}(FQ, X)$$

is bijective.
Then there is a unique way to extend the assignment $X \mapsto U_X$ to a functor $G \colon \mathcal{D} \to \mathcal{C}$ such that G is right adjoint to F with counit $\epsilon \colon FG \to \mathbb{1}_{\mathcal{D}}$.

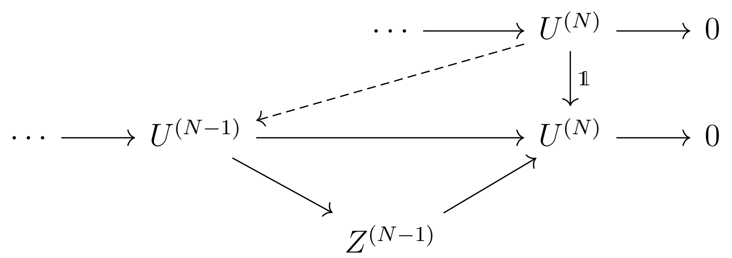
5.3. G and F are mutually inverse

Before we can prove that F and G are mutually inverse, we need to understand under which conditions an object Z of $K^-(\Lambda{-}\mathrm{Proj})$ is killed by G. It will turn out, in fact, that this never happens unless Z is already zero.

Lemma 5.3.1. *Let Z be an object in $K^-(\Lambda{-}\mathrm{Proj})$. Then GZ is zero in the homotopy category $K^-(\mathrm{Sum}{-}T)$ if and only if*

$$\forall i \in \mathbb{Z} \colon \mathrm{Hom}_{K^-}(T, Z[i]) = 0.$$

Proof. The 'if' part follows directly from the construction of the T-resolution $GZ = U_Z$. Indeed, if $\mathrm{Hom}_{K^-}(T, Z[i])$ is always zero, then the construction dictates $N = 0$ and $Z^{(0)} = Z$. Inductively following the construction, we then see $U^{(i)} := \bigoplus_{T \to Z^{(i)}} T = 0$ and $Z^{(i-1)} = Z^{(i)} = Z$. In particular, we obtain $U = 0$.
For the converse, assume that GZ is contractible and let N be the largest index i such that $\mathrm{Hom}_{K^-}(T, Z[i])$ doesn't vanish (if it exists). Consider the following diagram where the dashed arrow is a component of the homotopy which contracts GZ:



We see from this diagram that the map $Z^{(N-1)} \to U^{(N)}$ is a split epimorphism. Hence, if we apply $\mathrm{Hom}(T, -)$ to the triangle

$$Z^{(N-1)} \to U^{(N)} \to Z^{(N)} \rightsquigarrow$$

we obtain an exact sequence

$$\mathrm{Hom}\left(T, Z^{(N-1)}\right) \to \mathrm{Hom}\left(T, U^{(N)}\right) \to \mathrm{Hom}\left(T, Z^{(N)}\right)$$

where the first map is surjective. Since the second map is surjective by construction, we get that $\mathrm{Hom}\left(T, Z^{(N)}\right) = 0$. As this equals $\mathrm{Hom}\left(T, Z[N]\right)$, we get a contradiction to our choice for N. $\square$

Note that so far all we ever used about the tilting complex T are the axiom T1 (which we used all the time) and the fact that $\mathbb{T} = \mathrm{Hom}_{K^b}(T, -)$ commutes with direct sums (which we used in Lemma 4.1.1), i.e. $T \in K^b(\Lambda{-}\mathrm{Proj})$ (using Theorem 2.1.1). The proof of the next proposition, which provides the final ingredient for the proof of the main theorem, will make use of condition T2 for the first and only time.

Proposition 5.3.2. *The functor* $G\colon K^-(\Lambda{-}\mathrm{Proj}) \longrightarrow K^-(\mathrm{Sum}{-}T)$ *is faithful on objects, i.e. does not send nonzero objects to zero.*

Proof. Let Z be an object in $K^-(\Lambda{-}\mathrm{Proj})$ with $GZ = 0$; we want to show that Z is already zero.
From Lemma 5.3.1 we get that T lies in the full subcategory $\mathrm{Ker}\, Z$ of $K^-(\Lambda{-}\mathrm{Proj})$, which consists of the objects that are killed by $\mathrm{Hom}\left(-, Z[i]\right)$ for all integers i. Since $\mathrm{Ker}\, Z$ is easily seen to be closed under direct sums and summands, it also contains $\mathrm{Add}{-}T$. Moreover, using the long exact Hom-sequence for triangles, one can check that $\mathrm{Ker}\, Z$ is triangulated; hence $K^b(\Lambda{-}\mathrm{Proj}) \subset \mathrm{Ker}\, Z$ by property T2. In particular $\mathrm{Ker}\, Z$ contains $\lceil \Lambda \rfloor$, i.e. every map $\lceil \Lambda \rfloor \to Z[i]$ is nullhomotopic. Maps of Λ-chain complexes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \Lambda & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow f & & \downarrow & & \\
\cdots & \longrightarrow & Z_{i-1} & \xrightarrow{\pm d_{i-1}} & Z_i & \xrightarrow{\pm d_i} & Z_{i+1} & \longrightarrow & \cdots
\end{array}$$

(where the sign in the lower row comes from the shift $[i]$ but is irrelevant) correspond precisely to elements $f(1) \in \mathrm{Ker}\,(d_i)$; they are nullhomotopic if and only if f factors through d_{i-1} if and only if $f(1)$ lies in $\mathrm{Im}\,(d_{i-1})$.
Hence $f \mapsto f(1)$ gives a bijection between $\mathrm{Hom}_{K^-}(\lceil \Lambda \rfloor, Z[i]) = 0$ and the the i-th homology of Z, which then has to vanish. Since, however, Z is a right-bounded complex of projective modules, $H_* Z = 0$ implies that Z is contractible. $\square$

And now, the stage is set for the

Proof of the Main Theorem 3.1.4. The statement about restrictions to subcategories is precisely Corollary 2.1.2, so we only need to prove that $F \dashv G$ form an adjoint equivalence of triangulated categories

$$K^-(\text{Sum}-T) \xleftrightarrow{\simeq} K^-(\Lambda-\text{Proj}) \ .$$

This proof is a purely formal argument which puts together everything we know about F and G and uses some abstract results that we shall cite but not prove.
Since we know that F is fully faithful by Theorem 4.5.1, a standard result from category theory [ML98, §IV.3, Theorem 1] tells us that the unit $\mathbb{1} \to GF$ must be a natural isomorphism; it is left to show that the counit $FG \to \mathbb{1}$ is invertible as well.
Fix any object X in $K^-(\Lambda-\text{Proj})$ and complete the counit map to a triangle

$$FGX \to X \to Z \rightsquigarrow \tag{5.13}$$

in $K^-(\Lambda-\text{Proj})$. Since an adjoint of a triangulated functor is triangulated [Nee01, Lemma 5.3.6], we can apply G to (5.13) and get a triangle

$$GFGX \to GX \to GZ \rightsquigarrow$$

in $K^-(\text{Sum}-T)$. More abstract nonsense tells us that $GFGX \to GX$ is an isomorphism if F is full [ML98, §IV.3, Exercise 6], so GZ must be zero by Fact B.3.2 (iv).
Since G is faithful on objects (Proposition 5.3.2), this implies that Z is already zero.
This concludes the proof of the theorem, since the triangle (5.13) forces $FGX \to X$ to be an isomorphism if Z vanishes (again by Fact B.3.2 (iv)). □

6. Application: Centers of derived equivalent rings are isomorphic

Theorem 6.1. *Let* Λ *and* Γ *be (bounded) derived equivalent rings, i.e. we have* $D^b(\Lambda) \simeq D^b(\Gamma)$ *as triangulated categories. Then* Λ *and* Γ *have isomorphic centers.*

Proof. Denote by $Z(\Lambda)$ and $Z(\Gamma)$ the centers of Λ and Γ respectively. Using the Main Theorem 3.1.4 fix some tilting complex T over Λ and identify $\Gamma \cong \mathrm{End}^{\mathrm{opp}}_{D^b(\Lambda)}(T)$.

We construct a homomorphism $\mu^T \colon Z(\Lambda) \to Z(\Gamma)$ as follows:
Take any element $\lambda \in Z(\Lambda)$ and let $\mu^T(\lambda)$ be the endomorphism of T given by multiplication by λ on each component T_i of T. The fact that this is indeed a Λ-endomorphism (and therefore an element of Γ) follows from the fact that λ is central in Λ. Now we claim that that $\mu^T(\lambda)$ is in $Z(\Gamma)$. This is true, since every (representative of an) endomorphism $\gamma \colon T \to T$ consists of Λ-linear maps on the components of T and therefore commutes with multiplication by the Λ-scalar λ, i.e. commutes with $\mu^T(\lambda)$.

Next, we consider the functor $G_T \colon D^b(\Lambda) \to D^b(\Gamma)$ as in the main theorem. Since G_T is a derived equivalence we have that $S := G_T\lceil\Lambda\rfloor$ is itself a tilting complex over Γ; therefore we get a map $\mu^S \colon Z(\Gamma) \to Z(\Lambda)$ as above.

Claim: μ^S is a left inverse to μ^T, i.e. $\mu^S \circ \mu^T = \mathbb{1}_{Z(\Lambda)}$.
The claim certainly implies the theorem since, by applying the same argument with S instead of T, we get that $\mu^S \colon Z(\Gamma) \to Z(\Lambda)$ has both a right inverse μ^T and a left inverse $\mu^{G_S\lceil\Gamma\rfloor}$ and thus is an isomorphism.

To prove the claim let $\lambda \in \Lambda$ be given. After identifying Λ with $\mathrm{End}^{\mathrm{opp}}_{D^b(\Gamma)}(S)$ we see that $\mu^S\left(\mu^T(\lambda)\right)$ is nothing but 'multiplication by $\mu^T(\lambda) \in \Gamma$', i.e. 'multiplication by 'multiplication by λ''. So, to prove the claim, we now have to actually analyze all the identifications and see that this corresponds to λ itself.
The ring Λ is identified with $\mathrm{End}^{\mathrm{opp}}_{D^b(\Gamma)}(S)$ as follows:

$$\begin{array}{rcl} \Lambda & \xrightarrow{\cong} & \mathrm{End}^{\mathrm{opp}}_{D^b(\Lambda)}(\lceil\Lambda\rfloor) \xrightarrow{G_T} \mathrm{End}^{\mathrm{opp}}_{D^b(\Gamma)}(G_T\lceil\Lambda\rfloor) = \mathrm{End}^{\mathrm{opp}}_{D^b(\Gamma)}(S) \\ \lambda & \mapsto & \text{'multiplication by } \lambda\text{'}. \end{array}$$

We need to remember at this point that $G_T\lceil\Lambda\rfloor$ is the complex $\mathrm{Hom}_{D^b(\Lambda)}\left(T, U_{\lceil\Lambda\rfloor}\right)$, where $U_{\lceil\Lambda\rfloor}$ is some T-resolution of $\lceil\Lambda\rfloor$. Moreover, when passing from $\lceil\Lambda\rfloor$ to $U_{\lceil\Lambda\rfloor}$, the endomorphism 'multiplication by λ' remains 'multiplication by λ', which then under the functor $\mathrm{Hom}_{D^b(\Lambda)}(T,-)$ becomes ‘concatenation with ‘multiplication by λ’’. The $\Gamma = \mathrm{End}^{\mathrm{opp}}_{D^b(\Lambda)}(T)$-action on $\mathrm{Hom}_{D^b(\Lambda)}\big(T, U_{\lceil\Lambda\rfloor}\big)$ is precisely given by concatenation, hence we see that λ gets identified with ‘multiplication by ‘multiplication by λ’’; this is exactly what we wanted. $\square$

7. Tilting complexes with Γ-action

We start this section by looking at an easy example of a tilting complex and analyze it in detail. This will provide the motivation for the results of Section 7.2.

7.1. Detailed discussion of a small example

Let Λ be the path algebra over some field k of the quiver $1 \underset{b}{\overset{a}{\rightleftarrows}} 2$ with the relation $ba = 0$. This means that Λ is just the k-vectorspace with basis $\{e_1, e_2, a, b, ab\}$ with the obvious multiplication.
Furthermore, consider the complex T of Λ-Λ-bimodules

$$\Lambda e_2 \otimes_k e_2\Lambda \longrightarrow \Lambda$$

concentrated in degrees 0 and 1, where the map is given by multiplication in Λ. Denote by $P_1 := \Lambda e_1 = k\langle e_1, a\rangle$ and $P_2 := \Lambda e_2 = k\langle e_2, b, ab\rangle$ the two standard projective summands of Λ. As a (left) Λ-module we can decompose $\Lambda e_2 \otimes_k e_2\Lambda$ as $(P_2 \otimes e_2) \oplus (P_2 \otimes a) \oplus (P_2 \otimes ab)$ and Λ as $P_2 \oplus P_1$. Therefore, as a complex of Λ(-left)-modules (forgetting the right Λ-action), T it is nothing but

$$P_2 \oplus P_2 \oplus P_2 \xrightarrow{\begin{pmatrix} \mathbb{1} & 0 & \beta\alpha \\ 0 & \alpha & 0 \end{pmatrix}} P_2 \oplus P_1, \tag{7.1}$$

where $\alpha\colon P_2 = \Lambda e_2 \to \Lambda e_1 = P_1$ and $\beta\colon P_1 = \Lambda e_1 \to \Lambda e_2 = P_2$ are just right-multiplication by a and b respectively.

The first thing we want to prove is that T, seen as an object of $K^-(\Lambda{-}\mathrm{Proj})$, is a tilting complex over Λ.

The complex T looks quite complicated in the form (7.1), so we want to simplify it a bit before continuing. Consider the two complexes T_1 and T_2 given as $P_2 \xrightarrow{\alpha} P_1$ and $P_2 \to 0$ respectively.
The canonical inclusion of $P_2 \xrightarrow{\mathbb{1}} P_2$ into T is a morphism of complexes with cokernel

$$P_2 \oplus P_2 \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} P_1,$$

which is nothing but $T_1 \oplus T_2$. Since $P_2 \xrightarrow{\mathbb{1}} P_2$ is contractible, we see that the canonical projection $T \to T_1 \oplus T_2$ is a homotopy equivalence; hence from now on we will identify T with $T_1 \oplus T_2$. Concretely, the homotopy equivalence looks as follows:

$$\begin{array}{ccc} P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} & P_1 \\ {\scriptstyle \begin{pmatrix} 0 & -\beta\alpha \\ \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}} \downarrow \uparrow {\scriptstyle \mathrm{pr}_{2,3}} & & \downarrow \uparrow {\scriptstyle \mathrm{pr}} \\ P_2 \oplus P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \mathbb{1} & 0 & \beta\alpha \\ 0 & \alpha & 0 \end{pmatrix}} & P_2 \oplus P_1 \end{array} . \tag{7.2}$$

Next, we check the axiom T1 for T, i.e. we consider morphisms $T \to T[n]$ for n some nonzero integer. Clearly the only nontrivial cases are $n = \pm 1$.
For the case $n = 1$ consider a map as in the diagram

$$\begin{array}{ccccc} P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} & P_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} f \\ f' \end{pmatrix}} & & \downarrow \\ 0 & \longrightarrow & P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} & P_1 \end{array} .$$

We see that $0 = f\alpha = f'\alpha$ as maps $P_2 \to P_2$. On the other hand, we know that f and f' are both scalar multiples of β, since there are no other maps $P_1 \to P_2$. However, $\beta\alpha$ is not zero, hence f and f' must be.
The case $n = -1$ is a bit trickier because not all maps of the form

$$\begin{array}{ccccc} 0 & \longrightarrow & P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} & P_1 \\ \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} f & f' \end{pmatrix}} & & \downarrow \\ P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} & P_1 & \longrightarrow & 0 \end{array}$$

are zero. It turns out, however, that they are nullhomotopic. To see this, observe that f and f' must be scalar multiples of α, say $\lambda\alpha$ and $\lambda'\alpha$ respectively. We can then use the following homotopy:

$$\begin{array}{ccccc} 0 & \longrightarrow & P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} & P_1 \\ \downarrow & {\scriptstyle \begin{pmatrix} 0 & \lambda' \\ 0 & 0 \end{pmatrix}} \swarrow & \downarrow {\scriptstyle \begin{pmatrix} f & f' \end{pmatrix}} & \swarrow {\scriptstyle \lambda} & \downarrow \\ P_2 \oplus P_2 & \xrightarrow{\begin{pmatrix} \alpha & 0 \end{pmatrix}} & P_1 & \longrightarrow & 0 \end{array} .$$

Before we look at the axiom T2', we define two maps $p\colon T_1 \to T_2$ and $\gamma\colon T_2 \to T_1$ by the diagrams

$$\begin{array}{ccc} P_2 & \xrightarrow{\alpha} & P_1 \\ \downarrow {\scriptstyle \mathbb{1}} & & \downarrow \\ P_2 & \longrightarrow & 0 \end{array} \quad \text{and} \quad \begin{array}{ccc} P_2 & \longrightarrow & 0 \\ \downarrow {\scriptstyle -\beta\alpha} & & \downarrow \\ P_2 & \xrightarrow{\alpha} & P_1 \end{array}$$

respectively. The reason for the sign in the definition of γ will become apparent later. To verify T2' we use the triangle coming from the short exact sequence

$$0 \to \lceil P_1 \rfloor_1 \to T_1 \xrightarrow{p} T_2 \to 0$$

to see that P_1 is in the triangulated subcategory $\langle \mathrm{Add}{-}T \rangle$ generated by $\mathrm{Add}{-}T$. This is enough, since $\lceil P_2 \rfloor = T_2$ is already in $\mathrm{Add}{-}T$ and so $\langle \mathrm{Add}{-}T \rangle$ contains both indecomposable projectives.

Now that we know that T is a tilting complex, we are interested in (the opposite of) its endomorphism ring (modulo homotopy); it will turn out that this is just the ring Λ itself. In other words, the functor F_T from the Main Theorem 3.1.4 is a nontrivial autoequivalence of $D^-(\Lambda)$.
We first analyze the strict endomorphism ring $\mathrm{End}_{C^-}(T)$, i.e. without considering homotopies, by looking at all the cases:

- A basis of the maps $T_2 \to T_2$ is formed by the identity, called i_2 and the map $-\beta\alpha\colon P_2 \to P_2$, which can be written as $p\gamma$.
- The space $\mathrm{Hom}_{C^-}(T_1, T_2)$ has a basis $\{p, p\gamma p\}$, since $\mathrm{Hom}(P_2, P_2)$ is spanned by $\mathbb{1}_{P_2}$ and $-\beta\alpha$.
- Morphisms $T_2 \to T_1$ are just maps $f\colon P_2 \to P_2$ with $\alpha f = 0$. Precisely $-\beta\alpha$ and its scalar multiples fulfil this condition, giving $\mathrm{Hom}_{C^-}(T_2, T_1) = k\langle\gamma\rangle$.
- Last but not least, we look at an arbitrary morphism $f\colon T_1 \to T_1$ given as:

$$\begin{array}{ccc} P_2 & \xrightarrow{\alpha} & P_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ P_2 & \xrightarrow{\alpha} & P_1 \end{array} .$$

  Write f_0 and f_1 as some linear combinations $\lambda_0 \cdot \mathbb{1}_{P_2} + \mu \cdot \beta\alpha$ resp. $f_1 = \lambda_1 \cdot \mathbb{1}_{P_1}$. The equation $\lambda_1 \cdot \alpha = f_1\alpha = \alpha f_0 = \lambda_0 \cdot \alpha$ gives then $\lambda_1 = \lambda_0$ and there is no restriction on μ. Hence $\mathrm{Hom}(T_1, T_1)$ has a basis given by $i_1 := \mathbb{1}_{T_1}$ (corresponding to $\lambda_0 = \lambda_1 = 1$, $\mu = 0$) and γp (corresponding to $\lambda_0 = \lambda_1 = 0$, $\mu = -1$).

We can therefore describe the strict endomorphism ring of T as the path algebra of the quiver $T_1 \underset{p}{\overset{\gamma}{\rightleftarrows}} T_2$ modulo the relation $\gamma p \gamma = 0$. As a k-vectorspace this is just $k\langle i_1, p, \gamma p, p\gamma p, i_2, \gamma, p\gamma \rangle$.
We would like to quotient out homotopy and then identify this endomorphism ring

with $\Lambda^{\text{opp}} \cong \text{End}_\Lambda({}_\Lambda\Lambda)$, which has the structure $P_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} P_2$ with $\alpha\beta = 0$, via $\gamma \mapsto \alpha$ and $p \mapsto \beta$.
First of all, we see that γp is indeed nullhomotopic by the homotopy

$$\begin{array}{ccc} P_2 & \xrightarrow{\alpha} & P_1 \\ {\scriptstyle -\beta\alpha}\downarrow & \swarrow{\scriptstyle -\beta} & \downarrow{\scriptstyle 0} \\ P_2 & \xrightarrow{\alpha} & P_1 \end{array}$$

Secondly, we observe that the two-dimensional k-vectorspace $\text{End}_{C^-}(T_2) = k\langle i_2, \gamma p\rangle$ does not become smaller when passing to the homotopy category, since T_1 is concentrated in a single degree and so it is unaffected by homotopies. Since γp isn't nullhomotopic, γ and p also survive when passing to the homotopy category.
Finally we have to check that $\mathbb{1}_{T_1}$ is not nullhomotopic. This is easy, since T_1 is obviously not acyclic, hence it can't be contractible.

Altogether we have proved that the composition

$$\begin{array}{rcccl} \Gamma := \text{End}^{\text{opp}}_{K^-}(T) = \text{End}^{\text{opp}}_{C^-}(T)/\gamma p & \longrightarrow & \text{End}^{\text{opp}}_\Lambda({}_\Lambda\Lambda) & \xleftarrow{\cong} & \Lambda \\ \gamma & \longmapsto & \alpha & \longleftarrow\!\shortmid & a \\ p & \longmapsto & \beta & \longleftarrow\!\shortmid & b \end{array}$$

is an isomorphism of k-algebras.
Recall that T (i.e. $\Lambda e_2 \otimes e_2\Lambda \to \Lambda$) is a complex of Λ-Λ-bimodules and so after identifying $\Gamma \cong \Lambda$ it also becomes a complex of Λ-Γ-bimodules.

7.2. Comparison of F_T with the tensor-with-T functor

We now set aside our specific example and move to the following more general situation: T is a tilting complex over some ring Λ with opposite endomorphism ring Γ and T is not only a complex of Λ-modules, but even of Λ-Γ-bimodules. In this setting there is always a canonical functor

$$D^-(\Gamma) \simeq K^-(\Gamma\text{-Free}) \longrightarrow K^-(\Lambda\text{-Proj}) \simeq D^-(\Lambda),$$

namely the functor $T \otimes^{\text{tot}} - = T \otimes^{\text{tot}}_\Gamma -$, which maps a complex X in $K^-(\Gamma\text{-Free})$ to the complex with entries

$$\left(T \otimes^{\text{tot}}_\Gamma X\right)_n = \bigoplus_{i+j=n} T_i \otimes_\Gamma X_j$$

and differentials

$$\sum_{i+j=n} t_i \otimes \mathbb{1}_{X_j} + (-1)^n \mathbb{1}_{T_i} \otimes_\Gamma d_j \colon \bigoplus_{i+j=n} T_i \otimes_\Gamma X_j \to \bigoplus_{i+j=n+1} T_i \otimes X_j,$$

where t and d are the differentials of T and X respectively.

However we already constructed a functor $F_T \colon D^-(\Gamma) \to D^-(\Lambda)$ in Section 4, hence a natural question arises:

Question 7.2.1. *Under which conditions are the functors F_T and $T \otimes_\Gamma^{\text{tot}} -$ isomorphic?*

There are some very stupid reasons how F_T and $T \otimes^{\text{tot}} -$ could fail to be isomorphic. For instance, let Λ be the polynomial ring $k[x]$ and let T just be the complex $\lceil {}_\Lambda\Lambda \rfloor$ with the regular representation sitting in degree zero. Furthermore, let $\Gamma := \operatorname{End}^{\text{opp}}(T) \cong \Lambda$ act on the only nontrivial component $T_0 = \Lambda$ of T from the right with x acting as zero. This makes T into a Λ-Γ-bimodule.
The functor $T \otimes_\Gamma^{\text{tot}} -$ is then not faithful, since it sends the map $\cdot x \colon \Gamma \to \Gamma$ to the zero map $T \to T$. Therefore $T \otimes_\Gamma^{\text{tot}} -$ cannot be isomorphic to F_T, which is an equivalence of categories.
Of course, the problem is that we have not put any restrictions on the action of Γ on the components of T so we can't even rule out that it is trival.

If we make some kind of compatibility assumption, we get the following interesting

Theorem 7.2.2. *Let Λ, T and Γ be as before i.e. let T be a complex of Λ-Γ-bimodules, which is tilting over Λ with opposite endomorphism ring Γ. Moreover, assume that the right action of Γ on the components of T is compatible with the identification $\Gamma = \operatorname{End}^{\text{opp}}_{K^-(\Lambda-\text{Proj})}(T)$ in the sense that the morphism*

$$\begin{aligned} \Gamma \longrightarrow \operatorname{End}^{\text{opp}}_{C^-(\Lambda)}(T) &\longrightarrow \operatorname{End}^{\text{opp}}_{K^-(\Lambda)}(T) = \Gamma \\ c \longmapsto{} & \cdot c \end{aligned} \tag{7.3}$$

induced by this action is just the identity map on Γ.
Then $T \otimes_\Gamma^{\text{tot}} -$ and F_T are isomorphic as functors $K^-(\Gamma\text{−Free}) \to K^-(\Lambda\text{−Proj})$.

Proof. First of all, we observe that the functor $T \otimes^{\text{tot}} -$ can be described on the level of strict complexes (i.e. forgetting the homotopies) as the composition

$$C^-(\Gamma\text{−Free}) \xrightarrow{C^-(T^\sharp \otimes_\Gamma -)} C^-\left(\text{Sum−}T^\sharp\right) \xrightarrow{\text{tot}} C^-(\Lambda\text{−Proj}), \tag{7.4}$$

where $T^\sharp$ is the strict complex underlying T and $\mathrm{Sum}{-}T^\sharp \subset C^b(\Lambda{-}\mathrm{Proj})$ is the full subcategory of sums of $T^\sharp$. Here

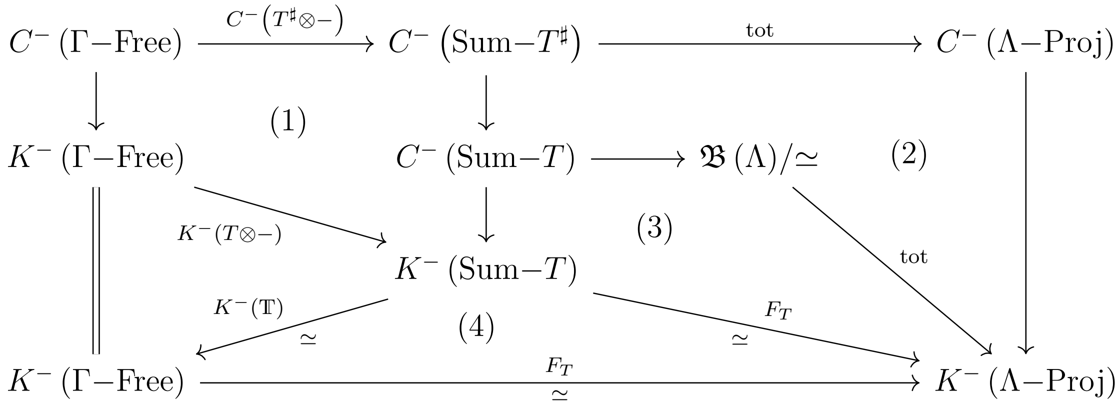
$$\mathrm{tot}\colon C^-\left(C^b(\Lambda{-}\mathrm{Proj})\right) \to C^-(\Lambda{-}\mathrm{Proj})$$

is the usual functor, which maps a complex X of bounded complexes X_i to the total complex with entries

$$(\mathrm{tot}\, X)_n \coloneqq \bigoplus_{i+j=n} (X_i)_j$$

and sums up the differentials of X and of the single X_i's (with some signs).

We can embed (7.4) into the big diagram



(7.5)

where we already know that the subdiagrams (1)-(4) are commutative:

(1) commutes by definition.

(2) commutes by Remark 4.4.1, because the objects $C^-\left(\mathrm{Sum}{-}T^\sharp\right)$ correspond to actual bicomplexes, i.e. objects of $\mathfrak{B}(\Lambda)$ with no differentials other than the zeroeth and first. This is true since maps in $\mathrm{Sum}{-}T^\sharp$ are actual maps of chain complexes and not homotopy classes and so the square of the homological differential of an object in $C^-\left(Sum - T^\sharp\right)$ is really zero and not just nullhomotopic.

(3) is just the defining property of F_T.

(4) commutes, since the two versions of F_T are in identified via $\mathbb{T}$ in Section 4.

Note that by definition $T\otimes^{\mathrm{tot}} -$ is the unique functor $K^-(\Gamma{-}\mathrm{Free}) \to K^-(\Lambda{-}\mathrm{Proj})$ which makes the outer rectangle of diagram (7.5) commutative. Hence to prove that F_T and $T\otimes^{\mathrm{tot}} -$ are the same, it is enough to see that the unnumbered part of diagram (7.5) commutes as well, i.e. that $K^-(T\otimes -)$ is an inverse to the equivalence of categories $K^-(\mathbb{T})$.

We shall now prove that $\mathbb{T} = \mathrm{Hom}_{K^-(\Lambda)}(T,-)$ and $T\otimes -$ are mutually inverse functors between Sum$-T$ and $\Gamma-$Free, which will finish the proof of Theorem 7.2.2. This is obvious on objects: for any index set I the complex $T\otimes_\Gamma \Gamma^{(I)}$ is nothing but $T^{(I)}$, which in turn gets mapped to $\Gamma^{(I)}$ by $\mathbb{T}$.
So let us now concentrate on the morphisms. The functor $T\otimes -$ induces the map

$$\begin{aligned} \mathrm{Hom}_\Gamma(\Gamma,\Gamma) &\rightarrow \mathrm{Hom}_{K^-}(T\otimes\Gamma, T\otimes\Gamma)\\ f &\mapsto \mathbb{1}_T\otimes f. \end{aligned}$$

If for f we take the right multiplication $\cdot c$ by some element $c\in\Gamma$ and identify $T\otimes\Gamma \xrightarrow{\cong} T$ (component-wise via $x\otimes\gamma\mapsto x\gamma$), then $\mathbb{1}_T\otimes(\cdot c)$ is simply given on the components of T by $\cdot c\colon T_i\to T_i$. Hence the composition

$$\begin{aligned} &\Gamma \xrightarrow{\cong} \mathrm{End}^{\mathrm{opp}}_\Gamma(\Gamma) \xrightarrow{T\otimes -} \mathrm{End}^{\mathrm{opp}}_{K^-}(T\otimes\Gamma) \cong \mathrm{End}^{\mathrm{opp}}_{K^-}(T) = \Gamma\\ &c \longmapsto \cdot c \end{aligned}$$

is nothing but the morphism (7.3), i.e. the identity by assumption. Moreover, in Lemma 4.1.1 we have already identified the opposite $\Gamma^{\mathrm{opp}}\to\mathrm{End}_\Gamma(\Gamma)$ of $c\mapsto\cdot c$ with the map $\mathrm{Hom}_{K^-}(T,T)\to\mathrm{Hom}_\Gamma(\Gamma,\Gamma)$ induced by $\mathbb{T}$.
All together we see that (up to the identification $T\otimes\Gamma\cong T$) the functors $\mathbb{T}$ and $T\otimes -$ are inverse to each other also on the basic Hom-sets $\mathrm{Hom}_{K^-}(T,T)$ resp. $\mathrm{Hom}_\Gamma(\Gamma,\Gamma)$.
Finally, we extend to the whole categories $\Gamma-$Free and Sum$-T$ by using the fact that both $T\otimes -$ and $\mathbb{T}$ commute with arbitrary direct sums. □

To conclude this section, we shall see that our motivating example of Section 7.1 fulfills the compatibility assumption of Theorem 7.2.2.
Keep in mind that we identified Γ and Λ via $\gamma\mapsto a$ and $p\mapsto b$. On the Λ-right-module $\Lambda e_2\otimes_k e_2\Lambda$, which as a Λ-left-module is identified with $(P_2\otimes e_2)\oplus(P_2\otimes a)\oplus(P_2\otimes ab)$, the right-multiplication by a just maps the e_2-component to the a-component and kills everything else. Similarly, $\cdot b$ maps a to ab and kills the rest. Therefore, when we look at T in the form (7.1), the componentwise right-

multiplication by b (resp. a) is

$$
\begin{array}{ccc}
P_2 \oplus P_2 \oplus P_2 & \longrightarrow & P_2 \oplus P_1 \\
\Big\downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \\
P_2 \oplus P_2 \oplus P_2 & \longrightarrow & P_2 \oplus P_1
\end{array}
\quad \text{resp.} \quad
\begin{array}{ccc}
P_2 \oplus P_2 \oplus P_2 & \longrightarrow & P_2 \oplus P_1 \\
\Big\downarrow \begin{pmatrix} 0 & 0 & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \\
P_2 \oplus P_2 \oplus P_2 & \xrightarrow{\alpha} & P_2 \oplus P_1
\end{array}
$$

It is easy to check that under the homotopy equivalence (7.2) these two maps correspond precisely to p and (due to the choice of sign in the definition) to γ, respectively. We have therefore proved that in our case the map of algebras (7.3) appearing in Theorem 7.2.2 maps γ to $\cdot a \simeq \gamma$ and p to $\cdot b \simeq p$, hence it is the identity on Γ as desired.

A. Glossary of notation

- Λ and Γ are unitary an associative (but not necessarily commutative) rings, $\mathcal{A}$ is an additive or abelian category, $\mathcal{T}$ and $\mathcal{S}$ are triangulated categories.
- $\mathbb{N}$ and $\mathbb{Z}$ are the natural numbers (with 0) and integers, respectively.
- All chain complexes are written with lower indices and differentials that raise the degree, e.g. $\cdots \xrightarrow{d_{-2}} X_{-1} \xrightarrow{d_{-1}} X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \cdots$.
- $\lceil M \rfloor_n$ is the complex concentrated in degree n with component M; a special case is $\lceil M \rfloor \coloneqq \lceil M \rfloor_0$.
- If X is some complex, then $X[n]$ denotes the complex shifted to the left by n, i.e. with components $X[n]_i = X_{i+n}$.
- If X is some complex, then $X\big|_n$ is the complex obtained by truncating X to the left at the place $-n$, i.e. $\left(X\big|_n\right)_i = 0$ for $i < -n$.
- If $f\colon X \to Y$ is a morphism of complexes, then $C(f)$ and $M(f)$ are the mapping cone and mapping cylinder, respectively.
- $\Lambda-$Mod is the category of (not necessarily finitely generated) left Λ-modules.
- $\Lambda-$mod $\subset \Lambda-$Mod is the full subcategory of *finitely generated* Λ-modules.
- $\Lambda-$Proj $\subset \Lambda-$Mod is the full subcategory of *projective* Λ-modules.
- $\Lambda-$Free $\subset \Lambda-$Proj is the full subcategory of *free* Λ-modules.
- $\Lambda-$proj $\subset \Lambda-$Proj is the full subcategory of *finitely generated* projective Λ-modules.
- $C^*(\mathcal{A})$ is the category of complexes with entries in $\mathcal{A}$, where the symbol $* \in \{+,-,b,\emptyset\}$ indicates boundedness conditions; we abbreviate $C(\mathcal{A}) = C^\emptyset(\mathcal{A})$.
- $K^*(\mathcal{A})$ is the category of $*$-bounded complexes modulo homotopy.
- $D^*(\mathcal{A})$ is the derived category, where $*$ indicates boundedness conditions on homology; we abbreviate $D^*(\Lambda) \coloneqq D^*(\Lambda-\mathrm{Mod})$.
- Add$-X$ is the smallest class of objects (in some additive category) containing X which is closed under *finite* direct sums and direct summands.
- Sum$-X$ is the full subcategory (of some additive category) consisting of *arbitrary* direct sums of copies of X. For any set I we write $X^{(I)}$ for $\bigoplus_I X$.

B. Generalities on chain complexes

In this section we work in any additive category $\mathcal{A}$. Some definitions and statements (mostly concerning homology) do not make sense in a category which is only additive; in those cases we will assume that $\mathcal{A}$ is abelian without explicit mention. Readers not familiar with additive or abelian categories may think of them just as subcategories of the category $\Lambda-$Mod for some ring Λ. The most important feature about abelian categories is that they give us the notion of an exact sequence.
Everything presented in this section is classical and can be found in the standard literature on homological algebra [Wei94, Chapters 1+3+10] [GM03, Chapters III+IV].

B.1. Chain complexes, homology and homotopy

Definition B.1.1. *A **(chain) complex** $X = (X, d)$ with entries in $\mathcal{A}$ is a $\mathbb{Z}$-indexed sequence $(X_i)_{i\in\mathbb{Z}}$ of objects of $\mathcal{A}$ together with arrows $d_i = d_i^X : X_i \to X_{i+1}$ such that $d_{i+1}d_i = 0$ for all $i \in \mathbb{Z}$. The d_i's are called **differentials** of X.*
*A morphism of chain complexes $f : X \to Y$ (also-called a **chain map**) is a collection of arrows $(f_i : X_i \to Y_i)_{i\in\mathbb{Z}}$ with the obvious commutativity relation $f_{i+1}d_i^X = d_i^Y f_i$.*

Chain complexes and chain maps form a category, which we denote by $C(\mathcal{A})$ or $C^{\emptyset}(\mathcal{A})$. Sometimes we are also interested in the full subcategories $C^-(\mathcal{A})$, $C^+(\mathcal{A})$ or $C^b(\mathcal{A})$ of $C(\mathcal{A})$ consisting of all **right-bounded**, resp. **left bounded**, resp. **bounded** complexes, i.e. of those complexes X with $X_i = 0$ for all $i \gg 0$, resp. all $i \ll 0$, resp. all $i \gg 0$ and $i \ll 0$. For the rest of the section, $* \in \{+, -, b, \emptyset\}$ will always be an arbitrary but fixed choice.
The range of indices where a complex X has nonzero entries, i.e. the set

$$\operatorname{supp} X := \{i \in \mathbb{Z} \mid \exists a, b \in \mathbb{Z} : a \le i \le b \text{ and } X_a, X_b \ne 0\},$$

is called the **support** of X.

Proposition B.1.2. *[Wei94, Theorem 1.2.3] The category $C^*(\mathcal{A})$ inherits an additive/abelian structure from $\mathcal{A}$.* □

There is the following two-dimensional analogue of the concept of a chain complex.

Definition B.1.3. *A **bicomplex** $X = (X, c, \delta)$ with entries in $\mathcal{A}$ consists of objects $X^{i,j}$ of $\mathcal{A}$ for all $i, j \in \mathbb{Z}$ and a pair of **differentials** $c_X^{i,j} : X^{i,j} \to X^{i+1,j}$, $\delta_X^{i,j} : X^{i,j} \to X^{i,j+1}$ with the following two conditions:*

- *the differentials c and δ commute with each other, i.e. $c^{i,j+1}\delta^{i,j} = \delta^{i+1,j}c^{i,j}$ and*
- *fixing either the first or the second index gives a complex, i.e. $c^{i+1,j}c^{i,j} = 0$ and $\delta^{i,j+1}\delta^{i,j} = 0$.*

Remark B.1.1. There is an obvious way to define morphisms of chain complexes and it is easy to see that the resulting category $C^2(\mathcal{A})$ is isomorphic to $C(C(A))$.

Let $C^2_{\text{fin}}(\mathcal{A}) \subset C^2(\mathcal{A})$ be the full subcategory consisting of all bicomplexes X such that for all n the set $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i+j=n, X^{i,j} \neq 0\}$ is finite. We get a canonical functor $\operatorname{tot}\colon C^2_{\text{fin}}(\mathcal{A}) \to C(\mathcal{A})$, which maps (X, c, δ) to the complex with entries $(\operatorname{tot} X)_n = \bigoplus_{i+j=n} X^{i,j}$ and differentials $d_n^{\operatorname{tot} X} = \sum_{i+j=n} c^{i,j} + (-1)^n \delta^{i,j}$.

The following class of chain complexes is particularly easy, important or boring depending on the perspective.

Definition B.1.4. *A complex (X, d) is called* ***acyclic*** *or* ***exact*** *if for each index i we have $\operatorname{Ker} d_i = \operatorname{Im} d_{i-1}$. The i-th* ***homology*** *of a chain complex is defined as $H_i X \coloneqq \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$.*

The homology is well defined since $d_i d_{i-1} = 0$ implies $\operatorname{Im} d_{i-1} \subseteq \operatorname{Ker} d_i$ (as subobjects of X_i) and measures how far a complex is from being acyclic.
Homology can be extended to maps between complexes in a straightforward manner. This gives rise to a collection of homology functors $H_i\colon C^*(\mathcal{A}) \to \mathcal{A}$. A morphism which induces an isomorphism in each degree of homology is called a **quasi-isomorphism** or **homology-isomorphism**.

The following definition provides an important class of examples where homology is not faithful on maps.

Definition B.1.5. *A* ***(chain) homotopy*** *between two chain maps $f, g\colon X \to Y$ is a collection of arrows $h_i\colon X_i \to Y_{i-1}$ such that $f_i - g_i = h_{i+1} d_i^X + d_{i-1}^Y h_i$ for all $i \in \mathbb{Z}$. Two morphisms that admit a homotopy between them are called* ***homotopic****, in symbols $f \simeq g$.*

Homotopy of chain maps is an equivalence relation compatible with addition and composition of morphisms. We obtain therefore a well-defined additive category $K^*(\mathcal{A})$ (the so-called **homotopy category**) with the same objects as $C^*(\mathcal{A})$ and arrows given by homotopy classes of chain maps. There is a canonical additive functor $C^*(\mathcal{A}) \to K^*(\mathcal{A})$ which is universal with the property of identifying homotopic maps.

Warning B.1.2. The category $K^*(\mathcal{A})$ is not abelian in general, even if $\mathcal{A}$ is abelian. We will see however, that it carries a good substitute structure, namely that of a triangulated category. One can prove that a category which is both abelian and triangulated has to be semisimple [GM03, Exercise IV.1, p. 250], i.e. all short exact sequences split. From this it follows that $K^*(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple.

Definition B.1.6. *Two complexes are called* ***homotopy equivalent****, if they are isomorphic in the homotopy category. A complex is called* ***contractible*** *or* ***split exact*** *if it is homotopy equivalent to the zero complex.*

One can check that homotopic chain maps induce the same map in every degree of homology and so the functors H_i factor through the homotopy category. In particular, this means that split exact complexes are exact, as suggested by the terminology.

Example B.1.7. *The complex* $\cdots \to X_{-1} \oplus X_0 \to X_0 \oplus X_1 \to X_1 \oplus X_2 \to \cdots$ *with differentials* $\begin{pmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{pmatrix}$ *is contractible.*

B.2. Mapping cones and shifted complexes

We present some important ways to make new chain complexes out of old ones.

Definition B.2.1. *For any chain complex* X *and integer* i*, let* $X[i]$ *denote the so-called* ***shifted complex*** *which has components* $X[i]_n \coloneqq X_{n+i}$ *and differentials*

$$d[i]_n \colon X[i]_n = X_{n+i} \xrightarrow{(-1)^i d_{n+i}} X_{n+1+i} = X[i]_{n+1}.$$

In particular, we obtain an invertible endofunctor $[1] \colon X \mapsto X[1]$ *of the category of complexes* $C^*(\mathcal{A})$ *called* ***suspension*** *or* ***shift****. It is easy to see that* $[1]$ *descends to an invertible endofunctor of* $K^*(\mathcal{A})$*.*

Let $f \colon X \to Y$ *be a map of chain complexes. Then we can construct two more chain complexes out of this datum:*

1. *The* ***mapping cone*** $C(f)$ *has components* $C(f)_n \coloneqq Y_n \oplus X_{n+1}$ *and differentials*

$$C(f)_n = Y_n \oplus X_{n+1} \xrightarrow{\begin{pmatrix} c_n & f_{n+1} \\ 0 & d[1]_n \end{pmatrix}} Y_{n+1} \oplus X_{n+2} = C(f)_{n+1},$$

where c *and* d *are the differentials of* Y *and* X*, respectively. Note also that* $d[1]_n$ *is nothing but* $-d_{n+1}$*.*

2. *The* ***mapping cylinder*** $M(f)$ *has components* $M(f)_n \coloneqq X_n \oplus Y_n \oplus X_{n+1}$ *and differentials*

$$M(f)_n = X_n \oplus Y_n \oplus X_{n+1} \xrightarrow{\begin{pmatrix} d_n & 0 & -\mathbb{1}_{X_{n+1}} \\ 0 & c_n & f_{n+1} \\ 0 & 0 & d[1]_n \end{pmatrix}} X_{n+1} \oplus Y_{n+1} \oplus X_{n+2} = M(f)_{n+1}.$$

It is an easy calculation that the square of the given differentials is zero, thus both constructions give well-defined chain complexes.

Readers familiar with the analogous constructions from algebraic topology will not be surprised by the following results.

Proposition B.2.2. *For any map $f\colon X \to Y$ of chain complexes let $C(f)$ and $M(f)$ denote its mapping cone and mapping cylinder, respectively. Then*

(i) The canonical map $Y \to C(f)$, as well as $i_0\colon X \to M(f)$ and $i_1\colon Y \to M(f)$, which are just given by the inclusion of the corresponding summands, are maps of complexes.

(ii) A map $g\colon Y \to Z$ factors through $Y \to C(f)$ if and only if gf is null-homotopic.

(iii) The map $f\colon X \to Y$ is a homotopy equivalence if and only if the corresponding mapping cone $C(f)$ is contractible.

(iv) The map $f\colon X \to Y$ is a quasi-isomorphism if and only if the corresponding mapping cone $C(f)$ is acyclic.

(v) Giving a homotopy between two maps $g_0, g_1\colon X \to Y$ is the same as giving a map $h\colon M(\mathbb{1}_X) \to Y$ such that $hi_0 = g_0$ and $hi_1 = g_1$.

(vi) The canonical projection $p\colon M(\mathbb{1}_X) \to X$ which on components is given by $(\mathbb{1}, \mathbb{1}, 0)$ is a homotopy inverse to both inclusions $i_0, i_1\colon X \to M(\mathbb{1}_X)$.

(vii) The canonical map $C(f) \to X[1]$ can be identified up to homotopy with the map $C(f) \to C(Y \to C(f))$.

(viii) The mapping cone and mapping cylinder are functorial in the sense that every commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

induces maps $C(f) \to C(f')$ and $M(f) \to M(f')$ while respecting compositions and identities.

(ix) Both the mapping cone and the mapping cylinder construction commute with the shift functor $[1]$.

Proof. The statements (i), (ii) and (v)–(ix) are easy calculations. Point (iv) is Corollary 1.5.4 in Weibel's book [Wei94]. Point (iii) is the most difficult but it follows from an abstract result about triangulated categories (Fact B.3.2 (iv)) once we know that the homotopy category of complexes is triangulated with triangles $X \xrightarrow{f} Y \to C(f) \rightsquigarrow$. □

B.3. Triangulated categories

The mapping cone and the shift functor are a very important part of the structure of the homotopy category, because they make it into a triangulated category.
The language of triangulated categories provides a convenient axiomatic framework for most of the work with homotopy categories and derived categories.

Definition B.3.1. *A* ***triangulated category*** *is an additive category $\mathcal{T}$ together with an additive autoequivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a class of so-called* ***(distinguished) triangles*** *of the form*

$$X \to Y \to Z \to \Sigma X \tag{B.1}$$

with objects and arrows in $\mathcal{T}$ satisfying certain axioms. A sequence (B.1) *which is not necessarily a triangle is sometimes called a* ***candidate triangle****; if it actually is a triangle, then it is often written as $X \to Y \to Z \rightsquigarrow$ for emphasis.*
Maps of (candidate) triangles are defined in the obvious way.

A comprehensive treatise on triangulated categories which includes a complete definition and all important properties of triangulated categories was written by Neeman [Nee01, Chapter 1]. We shall collect here without proof some facts about triangulated categories that are used throughout the thesis, mostly without explicit mention.

Facts B.3.2. *(i) The class of triangles is closed under isomorphisms of candidate triangles.*

(ii) A sequence $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle if and only if the rotated sequence $Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ (please note the sign change) is a triangle.

(iii) Every map $X \to Y$ can be extended to a triangle $X \to Y \to Z \rightsquigarrow$ and this extension is unique up to (non-canonical) isomorphism.

(iv) In a triangle $X \to Y \to Z \rightsquigarrow$, the map $X \to Y$ is an isomorphism if and only if $Z = 0$.

(v) The following commutative diagram with triangles for rows can always be com-

pleted to a map of triangles:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle \Sigma x} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array} \tag{B.2}$$

In general, this completion is not unique.

(vi) For any object $W \in \mathcal{T}$, applying the functors $\operatorname{Hom}(-, W)$ *and* $\operatorname{Hom}(W, -)$ *to a triangle* (B.1) *gives long exact sequences of abelian groups*

$$\cdots \to \operatorname{Hom}(Z, W) \to \operatorname{Hom}(Y, W) \to \operatorname{Hom}(X, W) \to \operatorname{Hom}\left(\Sigma^{-1} Z, W\right) \to \cdots$$

and

$$\cdots \to \operatorname{Hom}(W, X) \to \operatorname{Hom}(W, Y) \to \operatorname{Hom}(W, Z) \to \operatorname{Hom}(W, \Sigma X) \to \cdots$$

(vii) If two out of the three maps x, y and z in diagram (B.2) *are isomorphisms, then so is the third.* □

The following is the main example for us and the reason we use triangulated categories.

Example B.3.3. *[Wei94, Proposition 10.2.4 and Corollary 10.2.5] The additive category $K^*(\mathcal{A})$ together with the suspension $\Sigma := [1]$ is triangulated if we declare the triangles to be those candidate triangles that are isomorphic to the canonical mapping cone sequence*

$$X \xrightarrow{f} Y \to C(f) \to X[1] \tag{B.3}$$

for some map $f \colon X \to Y$ in $K^*(\mathcal{A})$.

Example B.3.4. *If $X = \cdots \to 0 \to X_{-n} \to X_{-n+1} \to \cdots$ is any left bounded complex, then we have the so-called* ***standard truncation triangle***

$$X\big|_{n-1} \to X \to \lceil X_{-n} \rfloor_{-n} \rightsquigarrow, \tag{B.4}$$

where $X\big|_{n-1}$ is the truncated complex $\cdots \to 0 \to X_{-n+1} \to X_{-n+2} \to \cdots$ and $\lceil X_{-n} \rfloor_{-n}$ consists only of the object X_{-n} sitting in degree $-n$. Indeed, X is just the mapping cone of the morphism $\lceil X_{-n} \rfloor_{-n}[-1] = \lceil X_{-n} \rfloor_{-n+1} \to X\big|_{n-1}$ induced by the differential d_n of X and so (B.4) *is just (a rotation of) a mapping cone sequence of the form* (B.3)*.*

Even when working inside $K^*(\mathcal{A})$ we will try to avoid the internal mechanics of chain complexes and use the general language of triangulated categories whenever we can get away with it. This will significantly reduce notational clutter and help to highlight what is essential to the argument.

Of course we have a notion of structure-preserving morphisms of triangulated categories.

Definition B.3.5. *An additive functor $F\colon \left(\mathcal{T}, \Sigma^{\mathcal{T}}\right) \to \left(\mathcal{S}, \Sigma^{\mathcal{S}}\right)$ between triangulated categories is called* ***triangulated*** *or* ***exact*** *if F maps triangles $X \to Y \to Z \rightsquigarrow$ in $\mathcal{T}$ to triangles $FX \to FY \to FZ \rightsquigarrow$ in $\mathcal{S}$ and commutes with the suspension, i.e. $F\Sigma^{\mathcal{T}} \cong \Sigma^{\mathcal{S}} F$ as functors $\mathcal{T} \to \mathcal{S}$.*
A full additive subcategory $\mathcal{C} \subseteq (\mathcal{T}, \Sigma)$ is called a ***triangulated subcategory*** *if it is closed under isomorphisms and suspension, and becomes a triangulated category when we restrict Σ and our choice of triangles to $\mathcal{C}$.*

B.4. Derived categories

It is easy to see that each split exact sequence

$$0 \to X \xrightarrow{i} Y \to Z \to 0$$

of chain complexes gives rise to a triangle $X \to Y \to Z \rightsquigarrow$ in the homotopy category. One could hope that this would also work for arbitrary exact sequences, but this is far from true in general.
We always have a canonical map

$$\begin{array}{ccccccc}
X & \xrightarrow{i} & Y & \longrightarrow & & & C(i) \\
\| & & \| & & & & \downarrow \\
X & \xrightarrow{i} & Y & \longrightarrow & Z & \xleftarrow{\cong} & X/Y
\end{array}$$

which is a quasi-isomorphism but non necessarily a homotopy equivalence.
We are not completely out of luck, though. If we restrict to the category $K^{-}(\mathcal{A}\text{–Proj})$ of right-bounded projective objects we get what we want:

Proposition B.4.1. *The canonical functor $C^{-}(\mathcal{A}\text{–Proj}) \to K^{-}(\mathcal{A}\text{–Proj})$ maps short exact sequences to triangles.*

The reason for this is that all short exact sequences of projective objects split. More generally, we have the following

Lemma B.4.2. *(i) Every right-bounded exact complex of projective objects is split exact.*

(ii) Every quasi-isomorphism between right-bounded complexes of projective objects is an isomorphism.

Proof. Using Proposition B.2.2 (iv) and (iii) we can reduce to check the first claim. One can construct the desired splitting by an easy induction starting from the last nonzero term on the right and repeatedly using the defining property of projective objects. □

We can regard acyclic complexes as those objects that "ought" to be contractible but sometimes aren't for lack of homotopies. It is often a good idea to move to a setting where complexes behave as they should.

Theorem & Definition B.4.3. *There is a (unique) triangulated category $D^*(\mathcal{A})$, called the* ***derived category*** *of $\mathcal{A}$, together with a triangulated functor $K^*(\mathcal{A}) \to D^*(\mathcal{A})$ that is universal with the property of mapping acyclic complexes to zero. Equivalently, $K^*(\mathcal{A}) \to D^*(\mathcal{A})$ can be defined as the universal triangulated functor which sends quasi-isomorphisms to honest isomorphisms.*

Proof. Theorem B.4.3 is a special case of the abstract construction known as Verdier localization [Nee01, Theorem 2.1.8] which provides an universal triangulated functor killing a fixed triangulated subcategory. □

Corollary B.4.4. *The composition $C^*(\mathcal{A}) \to K^*(\mathcal{A}) \to D^*(\mathcal{A})$ maps short exact sequences to triangles.* □

Lemma B.4.2 tells us that, while we work with right-bounded complexes of projective objects, nothing happens in the transition from the homotopy category to the derived category. Actually, an even stronger statement holds true in a broad class of interesting cases.

Proposition B.4.5. *[Wei94, Theorem 10.4.8] Assume that $\mathcal{A}$ has enough projectives. Then the canonical functor*

$$K^-(\mathcal{A}\text{–Proj}) \hookrightarrow K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$$

is an equivalence of triangulated categories. □

Remark B.4.1. There is a dual version of Proposition B.4.5, where projective objects are replaced by injective ones and $-$ by $+$.

Note that the image of $D^b(\mathcal{A}) \subset D^-(\mathcal{A})$ consists of those complexes in $K^-(\mathcal{A}\text{–Proj})$ that have bounded homology. We can therefore identify $D^b(\mathcal{A})$ with this subcategory of $K^-(\mathcal{A}\text{–Proj})$.

Please note that there are some set-theoretic issues with derived categories, since $D^* (\mathcal{A})$ need not have small Hom-sets in general. Using Proposition B.4.5 we are safe, however, as long as we stick to complexes of modules which are bounded to the right. The assumption on the existence of enough projectives is easily seen to be satisfied in the abelian category of modules over some ring.

C. Inverse limits and the Mittag-Leffler condition

An **inverse system** in some category is a diagram of the form $A^0 \leftarrow A^1 \leftarrow A^2 \leftarrow \cdots$ and its inverse limit (if it exists) is the universal object $\lim_{\leftarrow n} A^n$ equipped with maps

$$\begin{array}{ccccccc} A & & & & & & \\ \downarrow & \searrow & & \searrow & \cdots & & \\ A^0 & \longleftarrow & A^1 & \longleftarrow & A^2 & \longleftarrow & \cdots \end{array}$$

such that everything commutes. It is well-known that taking inverse limits is left exact and one is often interested in measuring its failure to be exact. For this purpose one introduces the so-called first right derived functor of $\varprojlim$, which is a functor

$$\varprojlim{}^1 \colon \mathbf{AbGrp}^{\leftarrow \mathbb{N}} \longrightarrow \mathbf{AbGrp}$$

from the category of inverse system of abelian groups to the category of abelian groups. The precise definition of $\varprojlim^1$ together with a proof that the higher derived functors vanish can be found in Weibel's book [Wei94, Chapter 3.5]. What is important for us is the following property of $\varprojlim$ and $\varprojlim^1$, which is a general property of derived functors:

Proposition C.1. *[Wei94, Corollary 3.5.4] For every short exact sequence*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of inverse systems of abelian groups there is a "long" exact $\varprojlim$-sequence.

$$0 \to \varprojlim A^\bullet \to \varprojlim B^\bullet \to \varprojlim C^\bullet \to \varprojlim{}^1 A^\bullet \to \varprojlim{}^1 B^\bullet \to \varprojlim{}^1 C^\bullet \to 0 \qquad \text{(C.1)}$$

□

It is often good to know that $\varprojlim^1$ of some system $A^\bullet$ vanishes, since then the long exact $\varprojlim$-sequence tells us that every short exact sequence starting in $A^\bullet$ will stay exact when passing to the inverse limit. We will now define a large class of inverse systems which turn out to have vanishing $\varprojlim^1$.

Definition C.2. *An inverse system $A^0 \leftarrow A^1 \leftarrow \cdots$ of abelian groups is said to be **Mittag-Leffler** if for every natural number n the descending sequence*

$$A^n \supseteq \operatorname{Im}\left(A^n \leftarrow A^{n+1}\right) \supseteq \operatorname{Im}\left(A^n \leftarrow A^{n+2}\right) \supseteq \operatorname{Im}\left(A^n \leftarrow A^{n+3}\right) \supseteq \cdots$$

of images of maps ending in A^n is eventually stationary.

Theorem C.3. *[Wei94, Proposition 3.5.7] If an inverse system $A^\bullet$ of abelian groups is Mittag-Leffler, then $\varprojlim^1 A^\bullet = 0$.* □

Using this theorem and some abstract diagram chasing, it is possible to prove criteria for when homology commutes with inverse limits. We will restrict ourselves to the following partial result:

Proposition C.4. *Let $A^0 \leftarrow A^1 \leftarrow \cdots$ be an inverse system in $C(\mathbf{AbGrp})$, i.e. an inverse system of chain complexes of abelian groups. Fix an integer i and assume that the component inverse system $A_i^0 \leftarrow A_i^1 \leftarrow \cdots$ is Mittag-Leffler.*
Then the canonical map in the $(i+1)$-th homology

$$H_{i+1} \lim_{\leftarrow n} A^n \longrightarrow \lim_{\leftarrow n} H_{i+1} A^n$$

(go to the proof of the proposition to see how the map arises) is a surjection.

Remark C.1. Note that for some fixed homological degree i, the corresponding components $A_i^0 \leftarrow A_i^1 \leftarrow \cdots$ form an inverse system of abelian groups. Hence it makes sense to talk about this system being Mittag-Leffler.

Proof. Let d^n and $e = \lim_{\leftarrow n} d^n$ be the differentials of A^n and $\lim_{\leftarrow n} A^n$ respectively. Consider the defining short exact sequence of the $(i+1)$-th homology group

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} d_i^n & \longrightarrow & \operatorname{Ker} d_{i+1}^n & \xrightarrow{\pi^n} & H_{i+1}A^n & \longrightarrow & 0 \\ & & {\scriptstyle d_i^n}\big\uparrow & \nearrow{\scriptstyle d_i^n} & & & & & \\ & & A_i^n & & & & & & \end{array} .$$

In the limit we obtain a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim\limits_{\leftarrow n} \operatorname{Im} d_i^n & \longrightarrow & \lim\limits_{\leftarrow n} \operatorname{Ker} d_{i+1}^n & \xrightarrow{\lim_{\leftarrow n} \pi^n} & \lim\limits_{\leftarrow n} H_{i+1}A^n & \longrightarrow & \lim\limits_{\leftarrow n}{}^1 \operatorname{Im} d_i^n \\ & & \big\uparrow & & \cong\big\uparrow & & \big\uparrow & & \\ & & \lim\limits_{\leftarrow n} A_i^n & \xrightarrow{e_i} & \operatorname{Ker} \lim\limits_{\leftarrow n} d_{i+1}^n & \longrightarrow & H_{i+1} \lim\limits_{\leftarrow n} A^n & \longrightarrow & 0 \end{array} , \tag{C.2}$$

where the second vertical map is an isomorphism since $\varprojlim$ is left exact, i.e. commutes with kernels. We know by assumption that $A_i^\bullet$ is Mittag-Leffler. Moreover, we have a surjection of inverse systems $d_i^\bullet \colon A_i^\bullet \to \operatorname{Im} d_i^\bullet$ and it is easy to see that the Mittag-Leffler condition is inherited by quotients. Therefore $\operatorname{Im} d_i^\bullet$ is Mittag-Leffler and so $\lim_{\leftarrow n}^1 \operatorname{Im} d_i^n$ vanishes by Theorem C.3. From the top exact row of diagram C.2 we can then deduce that $\lim_{\leftarrow n} \pi^n$ is surjective; hence the dashed arrow, which is the object of our interest, is a surjection as well. □

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